GRADUATE STUDENT SEMINAR : THE INVARIANT SUBSPACES PROBLEM

QUESTION FROM LINEAR ALGEBRA :

V = VECTOR SPACE (OVER $\mathbb{R}$ OR $\mathbb{C}$)

$A : V \to V$ A LINEAR TRANSFORMATION

DOES THERE EXIST A SUBSPACE W OF V THAT IS NONTRIVIAL ($W \neq \{0\}$ AND $W \neq V$) AND INVARIANT UNDER A ($AW \subseteq W$) ?

ANSWER IS "NO" IF

$\dim V = 1$: NO NONTRIVIAL SUBSPACES AT ALL

$\dim V = 2$: E.G., ROTATION BY $\frac{\pi}{2}$ IN $\mathbb{R}^2$

FOR $\dim V > 2$ (INCLUDING ∞), THE ANSWER IS "YES" :

THEOREM : LET V BE A VECTOR SPACE (OVER $\mathbb{R}$ OR $\mathbb{C}$) WITH $\dim V > 2$ (PERHAPS ∞) AND LET $A : V \to V$ BE A LINEAR TRANSFORMATION. THEN THERE IS A SUBSPACE W OF V SATISFYING

$$W \neq \{0\} \quad \text{AND} \quad W \neq V$$

AND

$$AW \subseteq W.$$

PROOF: SELECT SOME NONZERO $x_0 \in V$. CONSIDER THE SET

$$\{ x_0, Ax_0, A^2x_0, \dots \}$$

FIRST SUPPOSE THIS SET IS LINEARLY INDEPENDENT. DEFINE A LINEAR FUNCTIONAL α ON ITS SPAN AS FOLLOWS (NOTE: FOR NOTATIONAL SIMPLICITY WE WILL WRITE ELEMENTS OF THE SPAN AS $\sum_{n=0}^{\infty} a_n A^n x_0$, WHERE ALL BUT FINITELY MANY OF THE a_n ARE ZERO.)

$$\alpha\left(\sum_{n=0}^{\infty} a_n A^n x_0\right) = \sum_{n=0}^{\infty} a_n$$ (WELL-DEFINED BY LINEAR INDEPENDENCE)

LET

$$W = \ker \alpha = \alpha^{-1}(0).$$

THEN W IS A LINEAR SUBSPACE OF V AND IS INVARIANT UNDER A BECAUSE

$$x = \sum_{n=0}^{\infty} a_n A^n x_0 \in W \Rightarrow \alpha(Ax) = \alpha\left(A\left(\sum_{n=0}^{\infty} a_n A^n x_0\right)\right)$$
$$= \alpha\left(\sum_{n=0}^{\infty} a_n A^{n+1} x_0\right)$$
$$= \sum_{n=0}^{\infty} a_n = 0$$
$$\Rightarrow Ax \in W$$

$W \neq V$ BECAUSE $x_0 \notin W$ $(\alpha(x_0) = 1)$

$W \neq \{0\}$ BECAUSE $x_0 - Ax_0 \neq 0$ (BY LINEAR INDEPENDENCE),

BUT $x_0 - Ax_0 \in W$ BECAUSE $\alpha(x_0 - Ax_0) = 1 - 1 = 0$.

NEXT SUPPOSE $\{x_0, Ax_0, A^2x_0, \dots\}$ IS LINEARLY DEPENDENT. LET k BE THE SMALLEST NON-NEGATIVE INTEGER FOR WHICH $A^k x_0$ IS A LINEAR COMBINATION OF THE PREVIOUS $A^n x_0$, $n = 0, 1, \dots, k-1$. THEN

$$A^k x_0 = c_0 x_0 + c_1 A x_0 + \dots + c_{k-1} A^{k-1} x_0$$

AND

$$\{x_0, Ax_0, \dots, A^{k-1} x_0\}$$

IS LINEARLY INDEPENDENT. MOREOVER,

$$\text{SPAN}\,\{x_0, Ax_0, \dots, A^{k-1}x_0\} = \text{SPAN}\,\{x_0, Ax_0, A^2x_0, \dots\}$$

DENOTE THIS SPAN BY W.

W IS CLEARLY INVARIANT UNDER A AND $W \neq \{0\}$ BECAUSE $x_0 \in W$. IF $W \neq V$ WE ARE DONE. SHOULD IT BE THE CASE THAT $W = V$, THEN $\{x_0, Ax_0, \dots, A^{k-1}x_0\}$ IS A BASIS FOR V (SO, $k > 2$ BY ASSUMPTION). THEN, AS ABOVE, WE DEFINE A LINEAR FUNCTIONAL α ON V BY

$$\alpha\left(\sum_{n=0}^{k-1} a_n A^n x_0\right) = \sum_{n=0}^{k-1} a_n$$

AND $\ker \alpha$ IS A NONTRIVIAL INVARIANT SUBSPACE

EXERCISE : CHECK THIS OUT AND NOTE WHERE THE ARGUMENT WOULD FAIL IF $k = 1, 2$. $\square$

AT LEAST IN THE INFINITE-DIMENSIONAL CASE, THIS RESULT IS NOT VERY INTERESTING.

REASON : MOST "INTERESTING" INFINITE-DIMENSIONAL VECTOR SPACES COME EQUIPPED WITH A NORM RELATIVE TO WHICH THEY ARE COMPLETE AND A LINEAR SUBSPACE DOES NOT COUNT AS A "SUBSPACE" UNLESS IT IS ALSO COMPLETE.

A FEW DEFINITIONS :

X = VECTOR SPACE (ASSUMED INFINITE-DIMENSIONAL AND COMPLEX)

<u>NORM</u> : $x \in X \longrightarrow \|x\| \in \mathbb{R}$

(a) $\|x\| \geq 0$ AND $\|x\| = 0 \Leftrightarrow x = 0$

(b) $\|\lambda x\| = |\lambda| \, \|x\|$

(c) $\|x+y\| \leq \|x\| + \|y\|$

<u>METRIC</u> : $d(x,y) = \|y-x\|$ $\longrightarrow$ OPEN SET, CLOSED SET, CONVERGENT SEQUENCE, CAUCHY SEQUENCE, ...

<u>BANACH</u> <u>SPACE</u> : VECTOR SPACE X WITH NORM $\| \ \|$ THAT IS <u>COMPLETE</u> (CAUCHY SEQUENCES CONVERGE)

<u>BANACH</u> <u>SUBSPACE</u> : LINEAR SUBSPACE THAT IS CLOSED (AND THEREFORE ALSO COMPLETE)

EXAMPLES :

1. $C[0,1]$ = CONTINUOUS COMPLEX-VALUED FUNCTIONS ON $[0,1]$ WITH POINTWISE ALGEBRAIC OPERATIONS AND $\|f\| = \sup_{0 \le x \le 1} |f(x)|$.

2. ℓ^1 : ALL $\xi = (\xi_1, \xi_2, \cdots)$ FOR WHICH $\sum_{n=1}^{\infty} |\xi_n| < \infty$ WITH COORDINATEWISE ALGEBRAIC OPERATIONS AND $\|\xi\| = \sum_{n=1}^{\infty} |\xi_n|$.

3. $L^1[0,1]$ WITH LEBESGUE MEASURE (FOR THOSE WHO KNOW WHAT THIS MEANS) WITH POINTWISE ALGEBRAIC OPERATIONS AND $\|f\| = \int_{[0,1]} |f| \, d\mu$.

4. $L^2[0,1]$ WITH LEBESGUE MEASURE (FOR THOSE WHO KNOW WHAT THIS MEANS) WITH POINTWISE ALGEBRAIC OPERATIONS AND $\|f\| = \left(\int_{[0,1]} |f|^2 d\mu \right)^{\frac{1}{2}}$.

UNLIKE THE FINITE-DIMENSIONAL SITUATION, A LINEAR TRANSFORMATION $A : X \to Y$ BETWEEN TWO BANACH SPACES NEED NOT BE CONTINUOUS (WITH RESPECT TO THE METRICS $d_X(x_1, x_2) = \|x_2 - x_1\|_X$ AND $d_Y(y_1, y_2) = \|y_2 - y_1\|_Y$). THE CONTINUOUS ONES ARE EASY TO CHARACTERIZE, HOWEVER :

<u>THEOREM</u> : X, Y BANACH SPACES AND $A : X \to Y$ LINEAR. THEN THE FOLLOWING ARE EQUIVALENT :

(a) A IS CONTINUOUS.

(b) A IS CONTINUOUS AT $0 \in X$.

(c) THERE IS A CONSTANT $K \geq 0$ SUCH THAT

$$\|Ax\| \leq K \|x\|$$

FOR ALL $x \in X$ (FOR THIS REASON, CONTINUOUS LINEAR TRANSFORMATIONS ARE GENERALLY CALLED <u>BOUNDED</u> LINEAR TRANSFORMATIONS).

THE <u>NORM</u> OF A BOUNDED LINEAR TRANSFORMATION IS DEFINED BY

$$\|A\| = \inf \{ K : \|Ax\| \leq K \|x\| \ \forall x \in X \}.$$

THEN

$$\|Ax\| \leq \|A\| \|x\|.$$

WITH THIS NORM AND POINTWISE ALGEBRAIC OPERATIONS THE SET

$$\mathcal{B}(X,Y) = \text{ALL BOUNDED LINEAR TRANSFORMATIONS } A : X \to Y$$

BECOMES A BANACH SPACE.

SPECIAL CASE :

$$\mathcal{B}(X) = \mathcal{B}(X,X) = \text{BOUNDED } \underline{\text{OPERATORS}} \text{ ON } X .$$

EXAMPLES :

1. THE IDENTITY I AND ANY SCALAR MULTIPLICATION $x \to \lambda x$ $(\lambda \in \mathbb{C})$ ARE IN $\mathcal{B}(X)$ FOR ANY X.

2. $A : C[0,1] \to C[0,1]$ $\qquad (Af)(t) = t f(t)$

3. $A : L^1[0,1] \to L^1[0,1]$ $\qquad (Af)(t) = t f(t)$

4. $A : \ell^1 \to \ell^1$ $\qquad A\xi = (0, \xi_1, \xi_2, \cdots)$

   (CALLED THE SHIFT OPERATOR ON ℓ^1)

5. $A : L^2[0,1] \to L^2[0,1]$ $\qquad (Af)(t) = \int_{[0,1]} K(t,s) f(t) \, d\mu$

   WHERE $K \in L^2([0,1] \times [0,1])$

   (CALLED A FREDHOLM INTEGRAL OPERATOR)

HILBERT AND SCHMIDT "INVENTED" FUNCTIONAL ANALYSIS IN THEIR STUDY OF INTEGRAL EQUATIONS VIA FREDHOLM INTEGRAL OPERATORS. THESE HAVE THE FOLLOWING VERY SPECIAL PROPERTY :

> AN OPERATOR $K \in \mathcal{B}(X)$ IS COMPACT IF IT CARRIES BOUNDED SETS ONTO SETS WITH COMPACT CLOSURE.

NOTE : CLOSED, BOUNDED SETS IN AN INFINITE-DIMENSIONAL SPACE NEED NOT BE COMPACT.

AND NOW FOR OUR PROBLEM (WHICH GOES BACK AT LEAST TO VON NEUMANN AND PERHAPS TO BANACH):

INVARIANT SUBSPACES PROBLEM:

X = A BANACH SPACE (INFINITE-DIMENSIONAL AND COMPLEX)

$A : X \to X$ A BOUNDED LINEAR OPERATOR ON X

DOES THERE EXIST A CLOSED LINEAR SUBSPACE M OF X THAT IS NONTRIVIAL ($M \neq \{0\}$ AND $M \neq X$) AND INVARIANT UNDER A ($AM \subseteq M$).

A FEW SIMPLE OBSERVATIONS:

- IF A HAS AN EIGENVALUE λ, THEN

$$M_\lambda = \{x \in X : Ax = \lambda x\}$$

IS A CLOSED SUBSPACE OF X, $M_\lambda \neq \{0\}$ BY DEFINITION, $M_\lambda \neq X$ UNLESS A IS JUST A SCALAR MULTIPLICATION (IN WHICH CASE ANY CLOSED SUBSPACE OF X IS INVARIANT UNDER A) AND M_λ IS INVARIANT UNDER NOT ONLY A, BUT ALSO ANY OPERATOR B THAT COMMUTES WITH A:

$$x \in M_\lambda \Rightarrow A(Bx) = B(Ax) = B(\lambda x) = \lambda Bx$$
$$\Rightarrow Bx \in M_\lambda .$$

- NOT EVERY BOUNDED OPERATOR ON X HAS AN EIGENVALUE, E.G., THE SHIFT OPERATOR

$$\xi = (\xi_1, \xi_2, \dots) \in \ell^1 \longrightarrow A\xi = (0, \xi_1, \xi_2, \dots) \in \ell^1$$

- THERE IS AN "OBVIOUS" THING TO TRY: SELECT SOME NONZERO $x_0 \in X$ AND CONSTRUCT

$$\{x_0, Ax_0, A^2x_0, \dots\} \longrightarrow \text{SPAN}\{x_0, Ax_0, A^2x_0, \dots\}$$

$$\longrightarrow \text{CLOSURE IN } X \text{ OF SPAN}\{x_0, Ax_0, A^2x_0, \dots\}$$

RESULT IS A CLOSED LINEAR SUBSPACE OF X THAT IS INVARIANT UNDER A AND CERTAINLY $\neq \{0\}$.

PROBLEM : TAKING THE CLOSURE IS LIKELY TO YIELD SOMETHING QUITE LARGE, E.G., ALL OF X.

SOME HISTORY :

1. 1954 ARONSZAJN AND SMITH

THE ANSWER IS "YES" IF A IS A COMPACT OPERATOR.

2. 1966 BERNSTEIN AND ROBINSON

THE ANSWER IS "YES" IF SOME POLYNOMIAL IN A IS COMPACT.

3. 1973 LOMONOSOV

THE ANSWER IS "YES" IF A COMMUTES WITH SOME COMPACT OPERATOR.

<u>NOTE</u>: THIS IMPLIES BOTH OF THE PREVIOUS RESULTS, BUT LOMONOSOV'S PROOF IS <u>MUCH</u> SIMPLER THAN THOSE OF ARONSZAJN / SMITH AND BERNSTEIN / ROBINSON.

4. 1976 ENFLO

THE ANSWER IS "NO" IN GENERAL.

LOMONOSOV'S THEOREM IS ACTUALLY MUCH MORE GENERAL THAN THE STATEMENT ABOVE. A COMPLETE PROOF WILL BE GIVEN IN THE WEEKLY ANALYSIS SEMINAR.