

IMPLICIT DIFFERENTIATION

MOTIVATION : COMPARE

1.  $x^2 - y^3 = 3$  WITH  $y = \sqrt[3]{x^2 - 3}$

2.  $x^2 + y^2 = 1$  WITH  $y = \pm \sqrt{1 - x^2}$

3.  $x^3 + y^3 = 3xy$  WITH ?

y IS GIVEN IMPLICITLY  
AS A FUNCTION OF x

y IS GIVEN EXPLICITLY  
AS A FUNCTION OF x

NOTE THAT IT'S EASY TO COMPUTE  $y'$  DIRECTLY FROM  $y = \sqrt[3]{x^2 - 3}$  :

$$y' = ((x^2 - 3)^{\frac{1}{3}})' = \frac{1}{3} (x^2 - 3)^{-\frac{2}{3}} (2x)$$
$$= \frac{2x}{3(x^2 - 3)^{2/3}}$$

BUT ALSO NOTICE THAT YOU CAN GET  $y'$  DIRECTLY FROM  $x^2 - y^3 = 3$  BY REGARDING  $y$  AS SOME "UNKNOWN" FUNCTION OF  $x$ , DIFFERENTIATING BOTH SIDES AND USING THE CHAIN RULE :

$$(x^2 - y^3)' = (3)'$$
$$(x^2)' - (y^3)' = 0$$
$$2x - 3y^2 y' = 0 \quad (\text{CHAIN RULE!})$$
$$y' = \frac{2x}{3y^2}$$

THE "y" APPEARS HERE BECAUSE WE HAVEN'T SOLVED  $x^2 - y^3 = 3$  FOR y.

IF WE DID SOLVE FOR y WE'D GET  $y = (x^2 - 3)^{\frac{1}{3}}$  AND, SUBSTITUTING THIS, THE TWO ANSWERS FOR y' WOULD BE THE SAME.

SOMETIMES HAVING THE y IN THE EXPRESSION FOR y' IS A GOOD THING :

EXAMPLE : FIND THE SLOPE OF THE TANGENT LINE TO THE CIRCLE  $x^2 + y^2 = 1$  AT AN ARBITRARY POINT (a, b) (WITH  $b \neq 0$ )

NOTE : DON'T KNOW WHETHER (a, b) IS ON THE TOP HALF ( $y = \sqrt{1 - x^2}$ ) OR THE BOTTOM HALF ( $y = -\sqrt{1 - x^2}$ ).

FIND y' AS WE DID FOR  $x^2 - y^3 = 3$  :

$$(x^2 + y^2)' = 0$$

$$(x^2)' + (y^2)' = 0$$

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y} \quad (y \neq 0)$$

THUS, THE SLOPE AT (a, b) IS

$$y'(a, b) = -\frac{a}{b}$$

THIS PROCEDURE FOR FINDING  $y'$  IS EVEN MORE USEFUL WHEN IT'S SIMPLY IMPOSSIBLE TO SOLVE FOR  $y$  EXPLICITLY, E.G., FOR

$$x^3 + y^3 = 3xy$$

$$(x^3 + y^3)' = (3xy)'$$

$$(x^3)' + (y^3)' = 3(xy)'$$

$$3x^2 + 3y^2y' = 3(xy' + y(x)')$$

$$x^2 + y^2y' = xy' + y$$

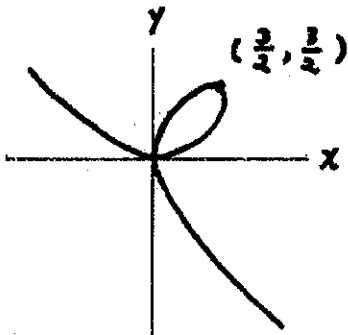
$$y^2y' - xy' = y - x^2$$

$$(y^2 - x)y' = y - x^2$$

$$y' = \frac{y - x^2}{y^2 - x}$$

MORE EXAMPLES :

1. THE GRAPH OF  $x^3 + y^3 = 3xy$  IS CALLED THE FOLIUM OF DESCARTES AND LOOKS LIKE THIS :



(A "FOLIUM" IS A LEAF AND DESCARTES IS SOME FRENCH GUY)

THE SLOPE OF THE TANGENT LINE AT  $(\frac{3}{2}, \frac{3}{2})$  IS

$$y'(\frac{3}{2}, \frac{3}{2}) = \frac{\frac{3}{2} - (\frac{3}{2})^2}{(\frac{3}{2})^2 - \frac{3}{2}} = -1$$

2. FIND  $y'$  IF  $y$  IS GIVEN IMPLICITLY BY  $x^2 = \frac{\cot y}{1 + \csc y}$

4.

$$(x^2)' = \left( \frac{\cot y}{1 + \csc y} \right)'$$

$$2x = \frac{(1 + \csc y)(-\csc^2 y)y' - (\cot y)(-\csc y \cot y)y'}{(1 + \csc y)^2}$$

$$2x = \frac{(-\csc^2 y - \csc^3 y + \csc y \cot^2 y)y'}{(1 + \csc y)^2}$$

$$2x = \frac{-\csc y (\csc y + \csc^2 y - \cot^2 y)y'}{(1 + \csc y)^2}$$

$$2x = \frac{-\csc y (\csc y + 1)y'}{(1 + \csc y)^2}$$

REMEMBER:  
 $\csc^2 y - \cot^2 y = 1$

$$2x = \frac{(-\csc y)y'}{1 + \csc y}$$

so  $y' = -\frac{2x(1 + \csc y)}{\csc y} = -2x(\sin y + 1)$

3. FIND  $y''$  IF  $x^3 + y^3 = 1$

$$(x^3 + y^3)' = (1)'$$

$$3x^2 + 3y^2 y' = 0$$

$$y' = -\frac{x^2}{y^2}$$

so

$$y'' = (y')' = \left( -\frac{x^2}{y^2} \right)' = -\frac{y^2(2x) - x^2(2yy')}{(y^2)^2}$$

$$= -\frac{2xy^2 - 2x^2y\left(-\frac{x^2}{y^2}\right)}{y^4} = -\frac{2xy^2 + 2x^4\left(\frac{1}{y}\right)}{y^4}$$

$$= -\frac{2xy^3 + 2x^4}{y^5}$$