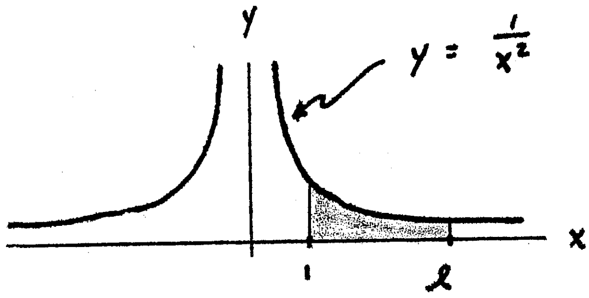
IMPROPER INTEGRALS : THESE COME IN SEVERAL FLAVORS :

CONSIDER, FOR EXAMPLE, $y = f(x) = \frac{1}{x^2}$



INTEGRATE $f(x)$ OVER $[1, \ell]$:

$$\int_1^\ell f(x)\,dx = \int_1^\ell \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^\ell$$

$$= -\frac{1}{\ell} - \left(-\frac{1}{1}\right)$$

$$= 1 - \frac{1}{\ell}$$ = AREA UNDER THE GRAPH FROM $x = 1$ TO $x = \ell$

NOW NOTICE THAT THIS APPROACHES 1 AS $\ell \to \infty$:

$$\lim_{\ell \to \infty} \int_1^\ell \frac{1}{x^2}\,dx = 1$$

TRYING THE SAME THING FOR $y = \frac{1}{x}$, HOWEVER, GIVES

$$\int_1^\ell \frac{1}{x}\,dx = \ln \ell$$

AND THIS $\to \infty$ AS $\ell \to \infty$.

WE WILL WRITE THESE LIMITS AS INTEGRALS WITH AN UPPER LIMIT OF INTEGRATION OF "∞" AND CALL THEM "IMPROPER INTEGRALS".

DEFINITION : THE IMPROPER INTEGRAL OF $f(x)$ OVER $[a,\infty)$ IS DEFINED TO BE

$$\int_a^\infty f(x)dx = \lim_{\ell\to\infty} \int_a^\ell f(x)dx$$

IF THE LIMIT EXISTS, $\int_a^\infty f(x)dx$ IS SAID TO CONVERGE. IF THE LIMIT DOES NOT EXIST, $\int_a^\infty f(x)dx$ IS SAID TO DIVERGE.

E.G., $\int_1^\infty \frac{1}{x^2}dx$ CONVERGES (AND ITS VALUE IS 1)

$\int_1^\infty \frac{1}{x}dx$ DIVERGES

MORE EXAMPLES :

1. $$\int_0^\infty \frac{1}{1+x^2}dx = \lim_{\ell\to\infty} \int_0^\ell \frac{1}{1+x^2}dx = \lim_{\ell\to\infty} \arctan x \Big|_0^\ell$$

$$= \lim_{\ell\to\infty} (\arctan \ell - \arctan 0) = \lim_{\ell\to\infty} \arctan \ell$$

$$= \frac{\pi}{2}$$ (SO THE INTEGRAL CONVERGES)

2. $$\int_1^\infty \frac{1}{x^p}dx$$

WE HAVE ALREADY SEEN THAT THIS DIVERGES WHEN $p = 1$ SO WE NOW ASSUME $p \neq 1$.

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{\ell\to\infty} \int_1^\ell x^{-p}\,dx = \lim_{\ell\to\infty} \frac{1}{-p+1}\, x^{-p+1}\Big|_1^\ell$$

$$= \frac{1}{1-p} \lim_{\ell\to\infty} (\ell^{1-p} - 1)$$

THIS LIMIT EXISTS ONLY WHEN $1-p<0$, I.E., $p>1$ (REMEMBER THAT WE ARE ASSUMING $p \neq 1$) AND IN THIS CASE $\ell^{1-p} \to 0$ AS $\ell \to \infty$ SO

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{1-p}(0-1) = \frac{1}{p-1} \quad \text{WHEN } p>1.$$

THUS,

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \begin{cases} \text{CONVERGES TO } \frac{1}{p-1} & \text{IF } p>1 \\ \text{DIVERGES IF } p \le 1 \end{cases}$$

3. $$\int_0^\infty (1-x)e^{-x}\,dx = \lim_{\ell\to\infty} \int_0^\ell (1-x)e^{-x}\,dx =$$

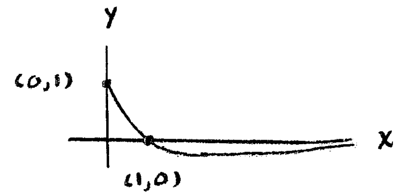
$$u = 1-x \qquad dv = e^{-x}dx$$
$$du = -dx \qquad v = -e^{-x}$$

$$\lim_{\ell\to\infty} \left[-(1-x)e^{-x}\Big|_0^\ell - \int_0^\ell e^{-x}\,dx \right] =$$

$$\lim_{\ell\to\infty} \left[(\ell-1)e^{-\ell} + 1 + e^{-x}\Big|_0^\ell \right] = \lim_{\ell\to\infty} \left[\frac{\ell-1}{e^\ell} + 1 + e^{-\ell} - 1 \right] =$$

$$\lim_{\ell\to\infty} \left[\frac{\ell-1}{e^\ell} + \frac{1}{e^\ell} \right] = 0+0 = 0$$ (1ST LIMIT BY ℓ'HÔPITAL'S RULE)

NOTE : HERE'S HOW THIS INTEGRAL ENDS UP BEING ZERO.



SIMILARLY, ONE DEFINES

$$\int_{-\infty}^{b} f(x)dx = \lim_{k \to -\infty} \int_{k}^{b} f(x)dx$$

(CONVERGES IF THE LIMIT EXISTS AND DIVERGES OTHERWISE)

E.G., $$\int_{-\infty}^{0} \frac{1}{1+x^2} dx = \lim_{k \to -\infty} \int_{k}^{0} \frac{1}{1+x^2} dx =$$

$$\lim_{k \to -\infty} \arctan x \Big|_{k}^{0} = \lim_{k \to -\infty} (\arctan 0 - \arctan k) =$$

$$\lim_{k \to -\infty} (-\arctan k) = -(-\tfrac{\pi}{2}) = \frac{\pi}{2}$$

(CONVERGES)

FINALLY, WE DEFINE

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$$

WHERE C IS ANY REAL NUMBER (USUALLY TAKEN TO BE 0).

$\int_{-\infty}^{\infty} f(x)dx$ IS SAID TO <u>CONVERGE</u> IF <u>BOTH</u> $\int_{-\infty}^{c} f(x)dx$ AND $\int_{c}^{\infty} f(x)dx$ CONVERGE AND TO <u>DIVERGE</u> OTHERWISE.

$$\text{E.G.,} \quad \int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \int_{-\infty}^{0} \frac{1}{1+x^2}dx + \int_{0}^{\infty} \frac{1}{1+x^2}dx$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

(CONVERGES).

<u>NOTE</u> : $\int_{-\infty}^{\infty} f(x)dx$ IS <u>NOT</u> THE SAME THING AS

$$\lim_{t\to\infty} \int_{-t}^{t} f(x)dx$$

FOR EXAMPLE, $\int_{0}^{\infty} x\,dx = \lim_{\ell\to\infty} \int_{0}^{\ell} x\,dx = \lim_{\ell\to\infty} \frac{1}{2}x^2\Big|_{0}^{\ell} =$

$\lim_{\ell\to\infty} \frac{1}{2}\ell^2$ OBVIOUSLY DIVERGES SO

$\int_{-\infty}^{\infty} x\,dx$ DIVERGES.

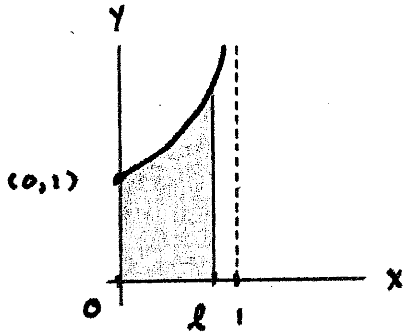
HOWEVER,

$$\lim_{t\to\infty} \int_{-t}^{t} x\,dx = \lim_{t\to\infty} \frac{1}{2}x^2\Big|_{-t}^{t} =$$

$$\lim_{t\to\infty} \left(\frac{1}{2}t^2 - \frac{1}{2}(-t)^2\right) = \lim_{t\to\infty} 0 = 0$$

THESE ALL COUNT AS THE FIRST "TYPE" OF IMPROPER INTEGRAL (INFINITE LIMITS OF INTEGRATION). NOW WE TURN TO THE SECOND "TYPE".

CONSIDER, FOR EXAMPLE, $f(x) = \frac{1}{\sqrt{1-x}}$ ON $[0,1]$.



NOT CONTINUOUS ON $[0,1]$ SO "$\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$" HAS NOT YET BEEN DEFINED.

HOWEVER, FOR ANY ℓ IN $0<\ell<1$,

$$\int_0^\ell \frac{1}{\sqrt{1-x}}\,dx = \int_0^\ell (1-x)^{-\frac{1}{2}}dx = -2\sqrt{1-x}\,\Big|_0^\ell$$

$$= -2\sqrt{1-\ell} + 2$$

AND

$$\lim_{\ell\to 1^-} \int_0^\ell \frac{1}{\sqrt{1-x}}\,dx = \lim_{\ell\to 1^-} (-2\sqrt{1-\ell}+2) = 2.$$

WE WILL DEFINE "$\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$" TO BE THIS LIMIT. MORE GENERALLY,

DEFINITIONS : IF $f(x)$ IS CONTINUOUS ON $[a,b)$, BUT $f(x) \to \pm\infty$ AS $x \to b^-$, THEN THE IMPROPER INTEGRAL OF $f(x)$ OVER $[a,b]$ IS

$$\int_a^b f(x)dx = \lim_{\ell \to b^-} \int_a^\ell f(x)dx .$$

IF THE LIMIT EXISTS, THEN $\int_a^b f(x)dx$ IS SAID TO CONVERGE ; OTHERWISE IT IS SAID TO DIVERGE.

E.G., $\int_0^1 \frac{1}{\sqrt{1-x}}dx$ CONVERGES (TO 2)

SIMILARLY,

$f(x)$ CONTINUOUS ON $(a,b$, BUT $f(x) \to \pm\infty$ AS $x \to a^+$

$$\int_a^b f(x)dx = \lim_{k \to a^+} \int_k^b f(x)dx$$

$f(x)$ CONTINUOUS ON $[a,b]$ EXCEPT AT $x = c$ IN (a,b) WHERE IT BLOWS UP, THEN

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

NOTE : BOTH OF THESE ARE IMPROPER INTEGRALS AND $\int_a^b f(x)dx$ CONVERGES IF AND ONLY IF THEY BOTH CONVERGE.

<u>NOTE</u> : ANY COMBINATION OF THE ABOVE TYPES IS ALSO POSSIBLE, E.G.,

$$\int_0^\infty \frac{1}{\sqrt{x}\,(x+1)}\,dx$$

IS IMPROPER FOR TWO REASONS (UPPER LIMIT ∞ AND $\frac{1}{\sqrt{x}\,(x+1)}$ BLOWS UP AT $x = 0$).

<u>EXAMPLES</u> :

1. $$\int_1^2 \frac{dx}{1-x} = \lim_{k\to 1^+} \int_k^2 \frac{1}{1-x}\,dx = \lim_{k\to 1^+} -\ln|1-x|\,\Big|_k^2 =$$

$$\lim_{k\to 1^+} \left(-\ln|1-2| + \ln|1-k|\right) =$$

$$\lim_{k\to 1^+} \ln|1-k|$$ WHICH DOES NOT EXIST SO

THE INTEGRAL DIVERGES.

2. $$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 (x-2)^{-\frac{2}{3}}\,dx + \int_2^4 (x-2)^{-\frac{2}{3}}\,dx$$

$$= \lim_{\ell\to 2^-} \int_1^\ell (x-2)^{-\frac{2}{3}}\,dx + \lim_{k\to 2^+} \int_k^4 (x-2)^{-\frac{2}{3}}\,dx$$

$$= \lim_{\ell\to 2^-} 3(x-2)^{\frac{1}{3}}\Big|_1^\ell + \lim_{k\to 2^+} 3(x-2)^{\frac{1}{3}}\Big|_k^4$$

$$= \lim_{\ell\to 2^-} \left[3(\ell-2)^{\frac{1}{3}} + 3\right] + \lim_{k\to 2^+} \left[3\sqrt[3]{2} - 3(k-2)^{\frac{1}{3}}\right]$$

$$= [0+3] + [3\sqrt[3]{2} - 0]$$

$$= 3(1+\sqrt[3]{2})$$

3. $$\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,dx = \int_0^1 \frac{1}{\sqrt{x}(1+x)}\,dx + \int_1^\infty \frac{1}{\sqrt{x}(1+x)}\,dx$$

<u>NOTE</u>: 1 COULD BE REPLACED BY ANYTHING IN $(0,\infty)$.

<u>ASIDE</u>: LET'S DO THE INDEFINITE INTEGRAL FIRST.

$$\int \frac{1}{\sqrt{x}(1+x)} = \int \frac{1}{1+(\sqrt{x})^2}\frac{1}{\sqrt{x}}\,dx$$

$$= 2\int \frac{1}{1+(\sqrt{x})^2}\frac{1}{2\sqrt{x}}\,dx$$

$$u = \sqrt{x}$$
$$du = \frac{1}{2\sqrt{x}}\,dx$$

$$= 2\int \frac{1}{1+u^2}\,du = 2\arctan u + C$$

$$= 2\arctan\sqrt{x} + C$$

THUS,

$$\int_0^1 \frac{1}{\sqrt{x}(1+x)}\,dx = \lim_{k\to 0^+}\int_k^1 \frac{1}{\sqrt{x}(1+x)}\,dx$$

$$= \lim_{k\to 0^+} 2\arctan\sqrt{x}\,\Big|_k^1$$

$$= \lim_{k\to 0^+}(2\arctan 1 - 2\arctan\sqrt{k})$$

$$= 2\left(\frac{\pi}{4}\right) - 0 = \frac{\pi}{2}$$

$$\int_1^{\infty} \frac{1}{\sqrt{x}(1+x)}\,dx = \lim_{\ell \to \infty} \int_1^{\ell} \frac{1}{\sqrt{x}(1+x)}\,dx =$$

$$\lim_{\ell \to \infty} 2\arctan\sqrt{x}\,\Big|_1^{\ell} = \lim_{\ell \to \infty} (2\arctan\sqrt{\ell} - 2\arctan 1) =$$

$$2\left(\frac{\pi}{2}\right) - 2\left(\frac{\pi}{4}\right) = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

THUS,

$$\int_0^{\infty} \frac{1}{\sqrt{x}(1+x)}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

4. $$\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1-\tan x}\,dx = \lim_{\ell \to \frac{\pi}{4}^-} \int_0^{\ell} \frac{\sec^2 x}{1-\tan x}\,dx$$

ASIDE: $$\int \frac{\sec^2 x}{1-\tan x}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C$$

$u = 1-\tan x$

$du = -\sec^2 x\,dx$

$$= -\ln|1-\tan x| + C$$

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1-\tan x}\,dx = \lim_{\ell \to \frac{\pi}{4}^-} -\ln|1-\tan x|\,\Big|_0^{\ell} =$$

$$\lim_{\ell \to \frac{\pi}{4}^-} (-\ln|1-\tan \ell| + \ln|1-\tan 0|)$$

WHICH DOES NOT EXIST BECAUSE THE FIRST TERM APPROACHES $-\ln|1-1|$ SO

THE INTEGRAL DIVERGES.