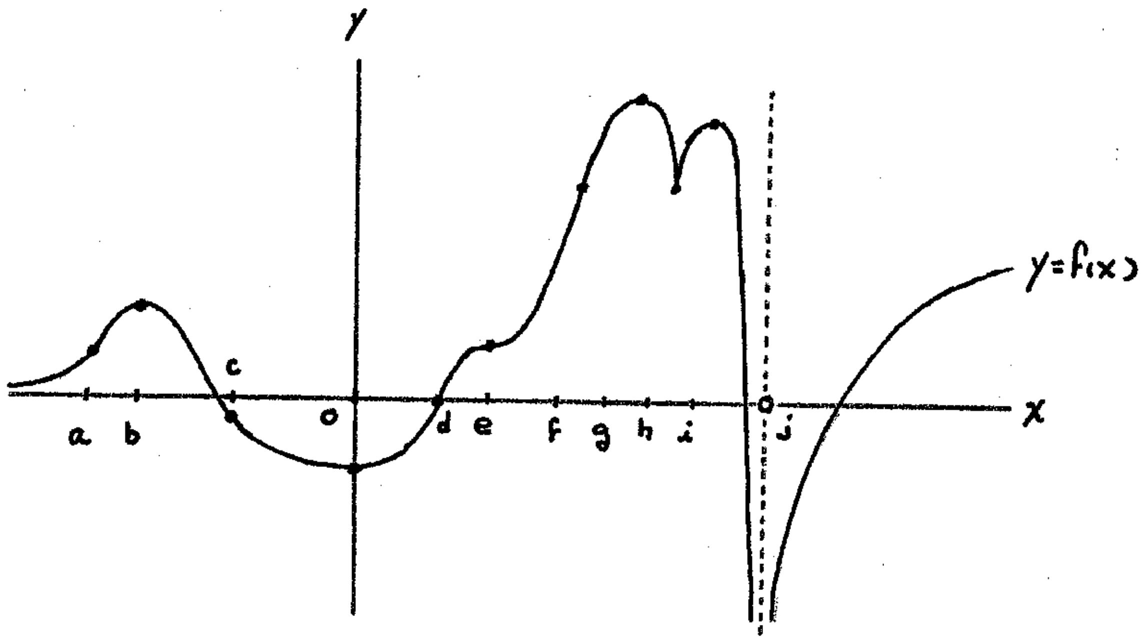
INCREASING, DECREASING, CONCAVITY



SOME TERMINOLOGY :

1. f IS INCREASING ON AN INTERVAL $\iff$ GRAPH IS RISING FROM LEFT TO RIGHT

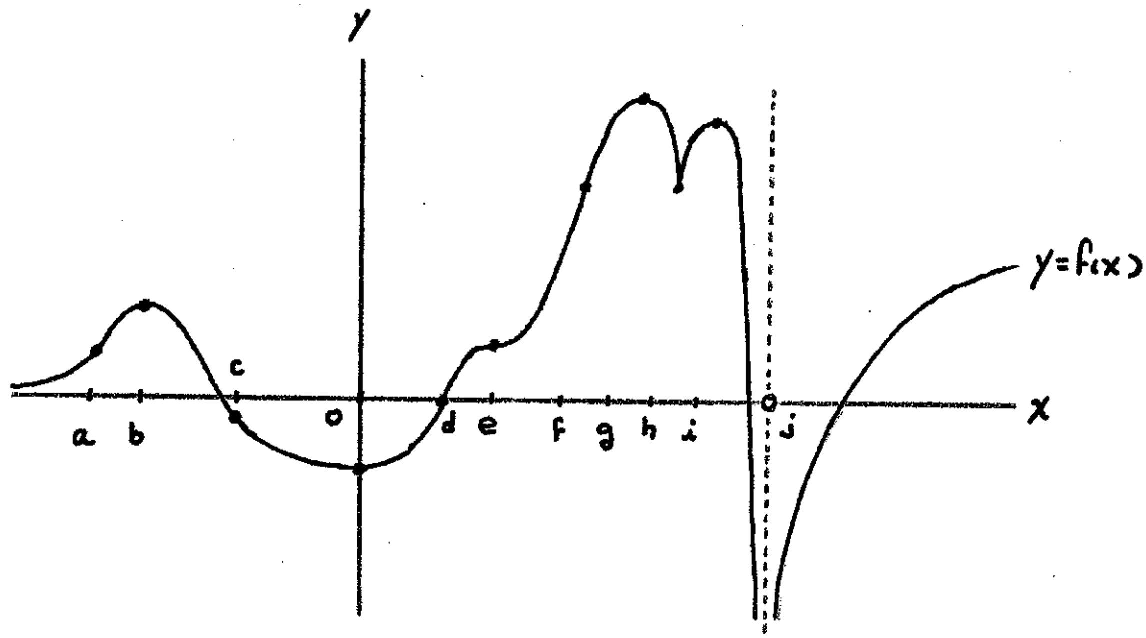
   $\iff$ $f'(x) > 0$ ON THAT INTERVAL

2. f IS DECREASING ON AN INTERVAL $\iff$ GRAPH IS FALLING FROM LEFT TO RIGHT

   $\iff$ $f'(x) < 0$ ON THAT INTERVAL

3. f CHANGES FROM INCREASING TO DECREASING AT A RELATIVE (OR LOCAL) MAXIMUM POINT.

4. f CHANGES FROM DECREASING TO INCREASING AT A RELATIVE (OR LOCAL) MINIMUM POINT.



5. f CAN INCREASE (OR DECREASE) IN TWO DIFFERENT WAYS :

f IS <u>CONCAVE</u> <u>UP</u> $\Leftrightarrow$ SLOPE OF THE TANGENT LINE IS INCREASING

$\Leftrightarrow$ $f'(x)$ IS INCREASING

$\Leftrightarrow$ $(f'(x))' > 0$

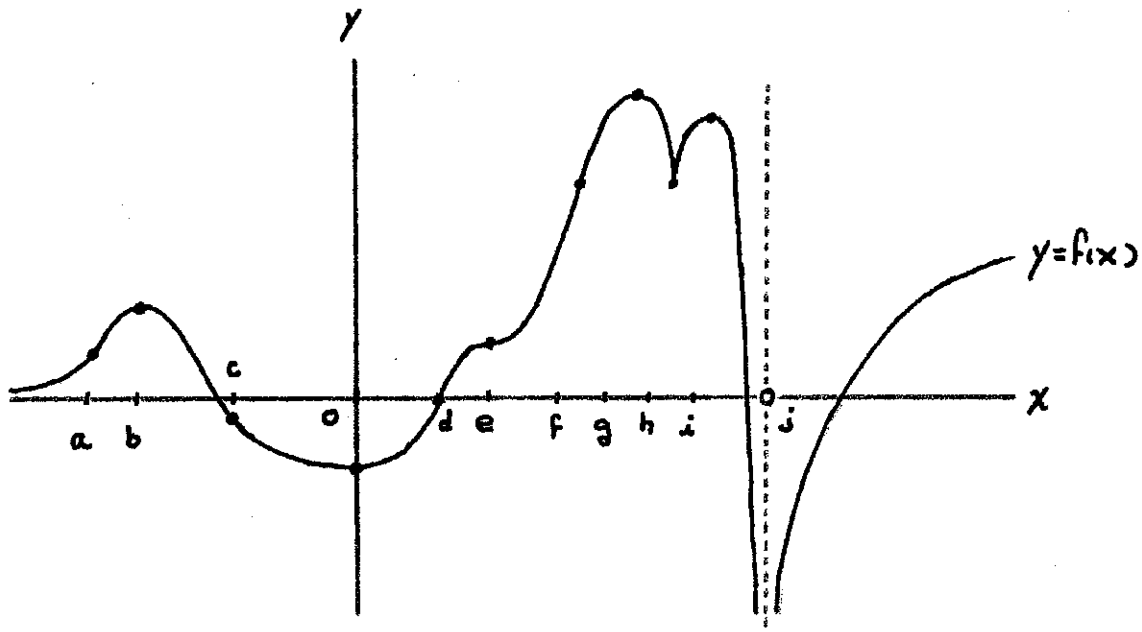
$\Leftrightarrow$ $f''(x) > 0$

f IS <u>CONCAVE</u> <u>DOWN</u> $\Leftrightarrow$ SLOPE OF THE TANGENT LINE IS DECREASING

$\Leftrightarrow$ $f'(x)$ IS DECREASING

$\Leftrightarrow$ $(f'(x))' < 0$

$\Leftrightarrow$ $f''(x) < 0$



6. A POINT AT WHICH f CHANGES FROM CONCAVE UP TO CONCAVE DOWN OR FROM CONCAVE DOWN TO CONCAVE UP IS CALLED AN <u>INFLECTION POINT</u>.

<u>NOTE</u> : AT A RELATIVE MAXIMUM POINT OR RELATIVE MINIMUM POINT, THE DERIVATIVE $f'(x)$ IS EITHER ZERO OR UNDEFINED (BUT THE DERIVATIVE MIGHT ALSO BE ZERO OR UNDEFINED AT A POINT THAT IS NOT A RELATIVE MAXIMUM OR MINIMUM). SIMILARLY, AN INFLECTION POINT ALWAYS OCCURS WHERE $f''(x)$ IS EITHER ZERO OR UNDEFINED (BUT $f''(x)$ MIGHT BE ZERO OR UNDEFINED AT A POINT THAT IS NOT AN INFLECTION POINT ALSO).

EXAMPLES: FOR EACH OF THE FOLLOWING FUNCTIONS f, FIND THE INTERVALS ON WHICH f IS INCREASING, DECREASING, CONCAVE UP AND CONCAVE DOWN AS WELL AS THE x-COORDINATES OF ANY RELATIVE MAXIMA, RELATIVE MINIMA AND INFLECTION POINTS.

1. $f(x) = x^4 - 2x^3$

$$f'(x) = 4x^3 - 6x^2 = 2x^2(2x-3)$$

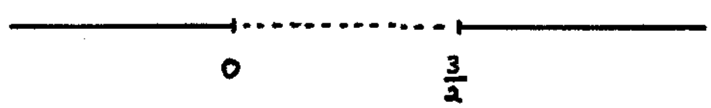
$$f''(x) = 12x^2 - 12x = 12x(x-1)$$

$f'(x)$ IS DEFINED EVERYWHERE AND IS ZERO ONLY WHERE

$$f'(x) = 0$$

$$2x^2(2x-3) = 0$$

$$x = 0,\ \tfrac{3}{2}$$



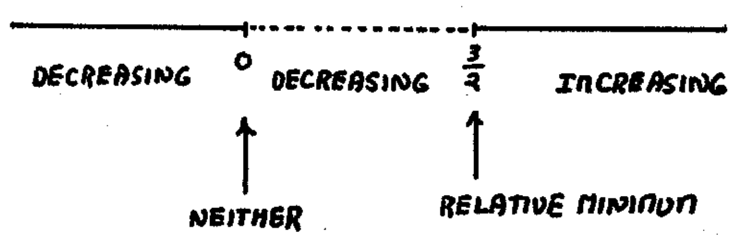
ON EACH OF THESE INTERVALS, $f'(x)$ CANNOT CHANGE SIGN. TEST A POINT IN EACH:

$x = -1$: $f'(-1) = 2(-1)^2(2(-1)-3) < 0$

$x = 1$: $f'(1) = 2(1)^2(2(1)-3) < 0$

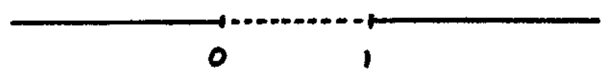
$x = 2$: $f'(2) = 2(2)^2(2(2)-3) > 0$

THUS,



$f''(x)$ IS DEFINED EVERYWHERE AND IS ZERO ONLY WHERE

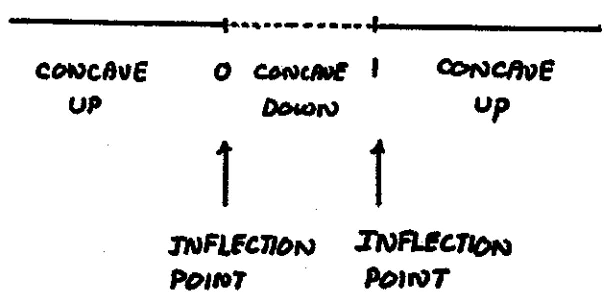
$$f''(x) = 0$$
$$12x(x-1) = 0$$
$$x = 0,\ 1$$



ON EACH OF THESE INTERVALS, $f''(x)$ CANNOT CHANGE SIGN. TEST A POINT IN EACH :

$$x = -1 \quad : \quad f''(-1) = 12(-1)(-1-1) > 0$$
$$x = \tfrac{1}{2} \quad : \quad f''(\tfrac{1}{2}) = 12(\tfrac{1}{2})(\tfrac{1}{2}-1) < 0$$
$$x = 2 \quad : \quad f''(2) = 12(2)(2-1) > 0$$

THUS,



<u>SUMMARY</u> : $f(x) = x^4 - 2x^3$ IS INCREASING ON $(\frac{3}{2}, \infty)$, DECREASING ON $(-\infty, 0) \cup (0, \frac{3}{2})$, CONCAVE UP ON $(-\infty, 0)$ AND $(1, \infty)$, CONCAVE DOWN ON $(0, 1)$, HAS A RELATIVE MINIMUM AT $x = \frac{3}{2}$ AND INFLECTION POINTS AT $x = 0$ AND $x = 1$.

2. $f(x) = xe^{-x}$

$$f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1-x)$$

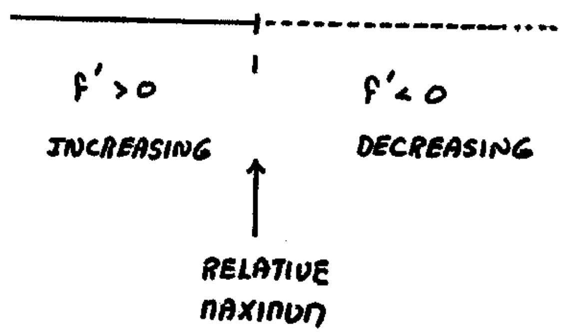
$$f''(x) = xe^{-x} - e^{-x} - e^{-x} = xe^{-x} - 2e^{-x} = e^{-x}(x-2)$$

BOTH $f'(x)$ AND $f''(x)$ ARE DEFINED EVERYWHERE.

$$f'(x) = 0$$

$$e^{-x}(1-x) = 0$$

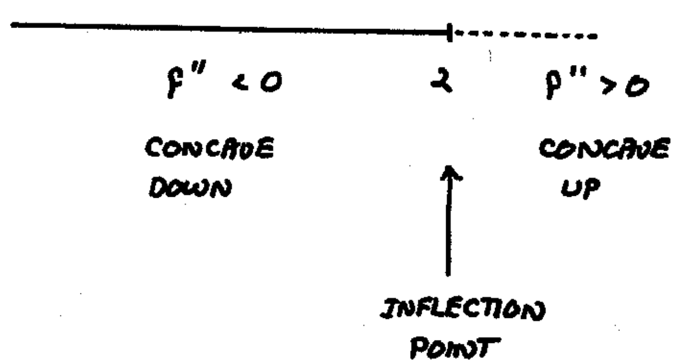
$x = 1$ (e^{-x} IS NEVER 0)



$$f''(x) = 0$$

$$e^{-x}(x-2) = 0$$

$x = 2$ (e^{-x} IS NEVER 0)



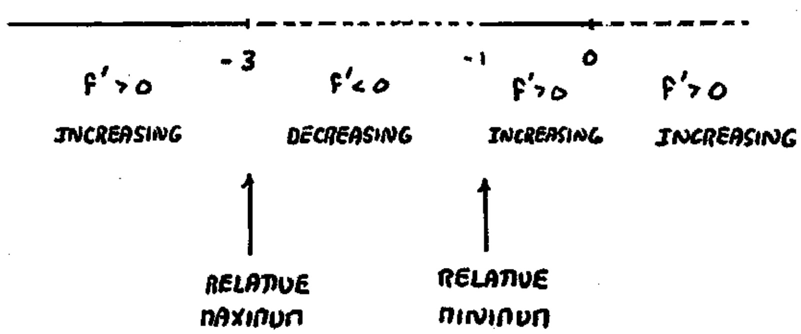
3. $f(x) = x^{\frac{1}{3}}(x+3)^{\frac{2}{3}}$

$$f'(x) = \frac{x+1}{(x^{\frac{2}{3}})(x+3)^{\frac{1}{3}}}$$

$$f''(x) = \frac{-2}{(x^{\frac{5}{3}})(x+3)^{\frac{4}{3}}}$$

I WILL LEAVE FOR YOU THE PLEASURE OF COMPUTING THESE DERIVATIVES.

$f'(x)$ IS UNDEFINED WHEN $X = 0$ AND $X = -3$ AND IT IS ZERO WHEN $X = -1$.



$f''(x)$ IS UNDEFINED AT $X = 0$ AND $X = -3$, BUT IS NEVER ZERO.