INFINITE SEQUENCES

EVENTUAL GOAL (FAR DOWN THE ROAD) : TO HAVE AN ENTIRELY NEW WAY OF LOOKING AT "FUNCTIONS", E.G., WE KNOW THAT, AT LEAST NEAR $x = 0$,

$$e^x \approx 1 + x$$

$$e^x \approx 1 + x + \frac{1}{2}x^2$$

$$e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

$$\vdots$$

$$e^x \approx \sum_{k=0}^{n} \frac{1}{k!} x^k$$

AND THAT THE APPROXIMATIONS GET BETTER AS YOU GO DOWN THE LIST. WE WOULD LIKE TO MAKE SOME SENSE OF "TAKING THE LIMIT" AND WRITING

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

BUT WE HAVE TO LEARN FIRST ABOUT "SEQUENCES" AND "SERIES".

SEQUENCES :

1. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

2. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

3. $1, -1, 1, -1, \ldots$

4. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

5. $3, 3, 3, 3, \ldots$

6. $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots$

7. $1, 3, 5, 7, \ldots$

OR, IN GENERAL,

$$a_1, a_2, a_3, a_4, \ldots$$

IT'S COMMON TO DESCRIBE A SEQUENCE BY JUST GIVING ITS $n^{\underline{TH}}$ <u>TERM</u> (ALSO CALLED <u>GENERAL TERM</u>):

$$\{a_n\}_{n=1}^{\infty}$$

E.G., FOR THE EXAMPLES ABOVE

1. $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \qquad a_n = \frac{1}{n}, \; n = 1, 2, 3, 4, \ldots$

2. $\frac{1}{2^0}, \frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots \qquad a_n = \frac{1}{2^{n-1}}, \; n = 1, 2, 3, 4, \ldots$

OR

$$a_n = \frac{1}{2^n}, \; n = 0, 1, 2, 3, \ldots$$

3. $(-1)^0, (-1)^1, (-1)^2, (-1)^3, \ldots \qquad a_n = (-1)^{n-1}, \; n = 1, 2, 3, 4, \ldots$

4. $\frac{1}{1+1}, \frac{2}{2+1}, \frac{3}{3+1}, \frac{4}{4+1}, \ldots$ $\quad a_n = \frac{n}{n+1}, \ n = 1, 2, \ldots$

5. $3, \ 3, \ 3, \ 3, \ldots$ $\quad a_n = 3, \ n = 1, 2, \ldots$

6. $(-1)^0 \frac{1}{1+1}, (-1)^1 \frac{2}{2+1}, (-1)^2 \frac{3}{3+1}, (-1)^4 \frac{4}{4+1}, \ldots$

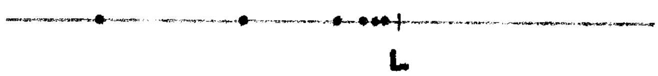
$$a_n = (-1)^{n-1} \frac{n}{n+1}, \ n = 1, 2, \ldots$$

7. $2(1)-1, \ 2(2)-1, \ 2(3)-1, \ 2(4)-1, \ldots$

$$a_n = 2n - 1, \ n = 1, 2, \ldots$$

SOME SEQUENCES "APPROACH" A NUMBER AS YOU MOVE OUT THE SEQUENCE (E.G., $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ "APPROACHES" 0) AND SOME DO NOT (E.G., $1, -1, 1, -1, \ldots$ OR $1, 3, 5, 7, \ldots$)

A SEQUENCE $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, a_4, \ldots\}$ IS SAID TO <u>CONVERGE</u> TO A NUMBER L (CALLED THE <u>LIMIT</u> OF THE SEQUENCE) IF ANY INTERVAL AROUND L (HOWEVER SMALL) CONTAINS ALL THE TERMS OF THE SEQUENCE BEYOND SOME POINT.



IN THIS CASE WE WRITE

$$L = \lim_{n \to \infty} a_n$$

IF NO SUCH NUMBER EXISTS WE SAY THAT THE SEQUENCE <u>DIVERGES</u>.

<u>EXAMPLES</u> :

1. $\lim_{n \to \infty} \frac{1}{n} = 0$

2. $\lim_{n \to \infty} \frac{1}{2^{n-1}} = 0$

3. $1, -1, 1, -1, \ldots$ DIVERGES

4. $\lim_{n \to \infty} \frac{n}{n+1}$

NOTICE THAT THIS LOOKS JUST LIKE A LIMIT WE COMPUTED A LONG TIME AGO

$$\lim_{x \to \infty} \frac{x}{x+1}$$

FOR WHICH WE EITHER (1) DIVIDED TOP AND BOTTOM BY x, OR (2) USED l'HÔPITAL'S RULE.

EVEN THOUGH A SEQUENCE IS NOT THE SAME THING AS A FUNCTION OF x THE BEHAVIOR IS THE SAME AND WE WILL NOT BOTHER TO CHANGE THE n s TO x s.

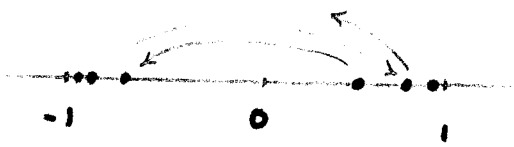
$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1$$

OR

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{(n)'}{(n+1)'} = \lim_{n\to\infty} 1 = 1$$

5. $\lim_{n\to\infty} 3 = 3$

6. $\left\{ (-1)^{n-1} \frac{n}{n+1} \right\}_{n=1}^{\infty}$ DIVERGES : WE SAW IN #4 THAT $\frac{n}{n+1} \to 1$, BUT HERE WE ALTERNATE THE SIGNS SO THE TERMS BOUNCE BACK AND FORTH, NEAR 1, THEN NEAR -1.



7. $\{ 2n-1 \}_{n=1}^{\infty}$ DIVERGES

8. $\left\{ (-1)^{n-1} \frac{1}{n^2} \right\}_{n=1}^{\infty} = \left\{ 1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \dots \right\}$

THIS ONE ALSO BOUNCES BACK AND FORTH ACROSS THE ORIGIN, BUT IT BOUNCES CLOSER TO 0 ON EITHER SIDE SO

$$\lim_{n\to\infty} (-1)^{n-1} \frac{1}{n^2} = 0$$

9. $\{3^n\}_{n=1}^{\infty} = \{3, 3^2, 3^3, \ldots\}$ CLEARLY DIVERGES, BUT

$\{(\frac{1}{2})^n\}_{n=1}^{\infty} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$ CONVERGES TO 0

<u>IN</u> <u>GENERAL</u>: $\{x^n\}_{n=1}^{\infty}$ CONVERGES TO 0 IF $-1 < x < 1$, CONVERGES TO 1 IF $x = 1$ AND DIVERGES FOR EVERY OTHER VALUE OF x.

10. $\left\{\frac{n!}{n^n}\right\}_{n=1}^{\infty}$ THIS TIME WE'LL USE THE <u>SQUEEZE</u> <u>THEOREM</u>.

$$0 \leq \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} = \frac{1}{n}\left(\frac{2 \cdot 3 \cdot \cdots n}{n \cdot n \cdots n}\right) \leq \frac{1}{n}$$

1. THIS GOES TO 0.

2. AND THIS GOES TO 0.

3. THIS IS TRAPPED BETWEEN 0 AND $\frac{1}{n}$ SO THE SQUEEZE THEOREM SAYS IT MUST GO TO 0 ALSO.