INFINITE SERIES

LET'S BUILD A SEQUENCE $\{S_0, S_1, S_2, \ldots\}$ IN THE FOLLOWING WAY :

$$S_0 = 1$$
$$S_1 = 1 + \tfrac{1}{2}$$
$$S_2 = 1 + \tfrac{1}{2} + \tfrac{1}{4}$$
$$S_3 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}$$
$$\vdots$$
$$S_n = \left(\tfrac{1}{2}\right)^0 + \left(\tfrac{1}{2}\right)^1 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \cdots + \left(\tfrac{1}{2}\right)^n$$
$$\vdots$$

SUMS OF MORE AND MORE POWERS OF $\frac{1}{2}$.

QUESTION : DOES THE SEQUENCE CONVERGE ? THAT IS, AS WE ADD TOGETHER MORE AND MORE POWERS OF $\frac{1}{2}$, DO THE SUMS "APPROACH" ANYTHING ?

THE ANSWER WOULD CERTAINLY BE "NO" IF WE REPLACED THE $\frac{1}{2}$ WITH A 1 SINCE THE SEQUENCE WOULD CLEARLY BLOW UP (I.E., $\rightarrow \infty$).

HOWEVER, OUR SEQUENCE ACTUALLY DOES CONVERGE BECAUSE

1. IT IS <u>INCREASING</u> ($S_{n+1} = S_n$ + A POSITIVE NUMBER).

2. <u>BOUNDED</u> <u>FROM</u> <u>ABOVE</u> BY 2 (GOING DOWN THE LIST FROM ONE TERM TO THE NEXT, WE ALWAYS ADD JUST HALF OF WHAT IT WOULD TAKE TO GET THE SUM UP TO 2).

THIS SHOULD BE A LITTLE SURPRIZING (ADDING MORE AND MORE POSITIVE NUMBERS, BUT THE SUMS DO NOT BLOW UP).

BUT WHAT DOES $\{ S_0, S_1, S_2, \ldots \}$ CONVERGE TO ?

USUALLY ANSWERING QUESTIONS LIKE THIS IS VERY HARD, BUT FOR THIS ONE THERE IS A TRICK THAT PROVIDES THE ANSWER :

TEMPORARILY CALL THE LIMIT OF $\{ S_n \}$ AS $n \to \infty$ JUST S.

$$S = \lim_{n \to \infty} S_n$$

NOW NOTICE THAT

$$S_n = 1 + \tfrac{1}{2} + (\tfrac{1}{2})^2 + \cdots + (\tfrac{1}{2})^n$$

SO

$$\tfrac{1}{2} S_n = \tfrac{1}{2} + (\tfrac{1}{2})^2 + \cdots + (\tfrac{1}{2})^n + (\tfrac{1}{2})^{n+1}$$

SUBTRACT THESE TWO TO GET

$$S_n - \tfrac{1}{2} S_n = 1 - (\tfrac{1}{2})^{n+1}$$

$$\tfrac{1}{2} S_n = 1 - (\tfrac{1}{2})^{n+1}$$

$$S_n = 2(1 - (\tfrac{1}{2})^{n+1}) = 2 - \frac{1}{2^n}$$

THUS,

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} (2 - \frac{1}{2^n}) = 2$$

<u>CONCLUSION</u> : AS YOU ADD TOGETHER MORE AND MORE POWERS OF $\frac{1}{2}$ (STARTING WITH $(\frac{1}{2})^0$) THE SUMS GET CLOSER AND CLOSER TO 2.

LET'S WRITE THIS OUT :

$$\lim_{n\to\infty} S_n = 2$$

$$\lim_{n\to\infty} (1 + \tfrac{1}{2} + \cdots + (\tfrac{1}{2})^n) = 2$$

$$\lim_{n\to\infty} \sum_{k=0}^{n} (\tfrac{1}{2})^k = 2$$

NOW WE INTRODUCE A <u>SHORTHAND</u> <u>NOTATION</u> FOR THIS :

$$\sum_{k=0}^{\infty} (\tfrac{1}{2})^k = 2$$

OR SOMETIMES

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots = 2$$

PLEASE UNDERSTAND THAT NEITHER OF THESE LAST TWO IS REALLY A SUM (YOU CAN'T ADD UP INFINITELY MANY NUMBERS, PERIOD!) THEY ARE SHORTHANDS FOR LIMIT STATEMENTS.

A SYMBOL LIKE

$$\sum_{k=0}^{\infty} a_k$$

OR

$$a_0 + a_1 + a_2 + \cdots$$

IS CALLED AN INFINITE SERIES (OR JUST SERIES). IT HAS A CORRESPONDING SEQUENCE OF PARTIAL SUMS

$$\begin{aligned} S_0 &= a_0 \\ S_1 &= a_0 + a_1 \\ S_2 &= a_0 + a_1 + a_2 \\ &\vdots \\ S_n &= a_0 + a_1 + a_2 + \cdots + a_n \\ &\vdots \end{aligned}$$

IF THIS SEQUENCE $\{S_n\}$ OF PARTIAL SUMS CONVERGES TO SOME REAL NUMBER S, THEN THE SERIES IS SAID TO CONVERGE TO S AND S IS CALLED THE SUM OF THE SERIES. IN THIS CASE WE WRITE

$$\sum_{k=0}^{\infty} a_k = S$$

OR

$$a_0 + a_1 + a_2 + \cdots = S.$$

IF THE SEQUENCE OF PARTIAL SUMS DOES NOT CONVERGE, THEN WE SAY THAT THE SERIES DIVERGES.

EXAMPLES:

1. THE SAME ARGUMENT GIVEN FOR $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2$

   PROVES THAT IF

$$\boxed{-1 < r < 1}$$

   THEN

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

   AND, IN FACT,

$$\boxed{\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}} \quad \text{(GEOMETRIC SERIES)}$$

   FOR ANY CONSTANT a, E.G.,

$$\sum_{k=0}^{\infty} \frac{3}{10^k} = \sum_{k=0}^{\infty} 3\left(\frac{1}{10}\right)^k = \frac{3}{1-\frac{1}{10}}$$

$$= \frac{3}{\frac{9}{10}} = \frac{10}{3}$$

2. $\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + \cdots$ DIVERGES BECAUSE ITS SEQUENCE OF PARTIAL SUMS IS

$$S_0 = 1$$
$$S_1 = 1-1 = 0$$
$$S_2 = 1-1+1 = 1$$
$$S_3 = 1-1+1-1 = 0$$
$$\vdots$$

AND THE SEQUENCE $\{1, 0, 1, 0, \ldots\}$ DIVERGES.

<u>NOTE</u> : THIS IS THE GEOMETRIC SERIES $(a=1)$ WHEN $r = -1$. IN FACT, THE GEOMERIC SERIES DIVERGES WHENEVER $r \leq -1$ OR $r \geq 1$.

3. $$\sum_{k=0}^{\infty} 3^k 5^{1-k} = \sum_{k=0}^{\infty} 3^k \cdot 5 \cdot 5^{-k} = \sum_{k=0}^{\infty} 5 \cdot 3^k \left(\frac{1}{5^k}\right)$$

$$= \sum_{k=0}^{\infty} 5\left(\frac{3}{5}\right)^k = \frac{5}{1-\frac{3}{5}} = \frac{5}{\frac{2}{5}}$$

$$= \frac{25}{2}$$

4. $$\sum_{k=1}^{\infty} \frac{5}{4^k} = \sum_{k=1}^{\infty} 5\left(\frac{1}{4}\right)^k$$

NOTE THAT THE INDEX k STARTS AT 1 THIS TIME. WE KNOW THAT

$$\sum_{k=0}^{\infty} 5\left(\frac{1}{4}\right)^k = \frac{5}{1-\frac{1}{4}} = \frac{5}{\frac{3}{4}} = \frac{20}{3}$$

$$5\left(\frac{1}{4}\right)^0 + \sum_{k=1}^{\infty} 5\left(\frac{1}{4}\right)^k = \frac{20}{3} \qquad (*)$$

$$\sum_{k=1}^{\infty} 5\left(\frac{1}{4}\right)^k = \frac{20}{3} - 5 = \frac{5}{3}$$

SO

$$\sum_{k=1}^{\infty} \frac{5}{4^k} = \frac{5}{3}$$

<u>NOTE</u> : THE LINE WITH THE (*) ABOVE IS ACTUALLY A LITTLE SLOPPY. WHY? AND HOW DO YOU JUSTIFY IT?

5. THE FORMULA FOR THE SUM OF A GEOMETRIC SERIES CAN BE USED TO WRITE RATIONAL NUMBERS GIVEN AS DECIMALS (I.E., $0.784784784\cdots$) AS QUOTIENTS OF INTEGERS.

$$\begin{aligned}
0.784784784\cdots &= 0.784 + 0.000784 + 0.000000784 + \cdots \\
&= 0.784(1) + 0.784(10^{-3}) + 0.784(10^{-6}) + \cdots \\
&= 0.784\left(\frac{1}{10^3}\right)^0 + 0.784\left(\frac{1}{10^3}\right)^1 + 0.784\left(\frac{1}{10^3}\right)^2 + \cdots
\end{aligned}$$

$$= \sum_{k=0}^{\infty} 0.784 \left(\frac{1}{10^3}\right)^k = \frac{0.784}{1-\frac{1}{10^3}} \; \frac{10^3}{10^3}$$

$$= \frac{784}{999}$$

GEOMETRIC SERIES ARE A RARE EXCEPTION TO A GENERAL RULE : OFTEN IT IS NOT SO HARD TO SHOW THAT A SERIES CONVERGES, BUT FINDING OUT WHAT IT CONVERGES TO IS <u>EXTREMELY</u> DIFFICULT.

THERE IS BASICALLY JUST ONE MORE EXCEPTION TO THIS RULE :

<u>TELESCOPING</u> <u>SERIES</u> :

CONSIDER THE SERIES $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right)$. WE'LL WRITE OUT ITS SEQUENCE OF PARTIAL SUMS :

$$S_1 = \frac{1}{1} - \frac{1}{2}$$

$$S_2 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right)$$

$$S_3 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$\vdots$$

$$S_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$\vdots$$

NOTICE THE CANCELING OF EVERY OTHER TERM. THE PARTIAL SUMS COLLAPSE (" TELESCOPE ") TO GIVE

$$S_n = 1 - \frac{1}{n+1}$$

AND THIS IS EASY TO TAKE THE LIMIT OF :

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \lim_{n \to \infty} S_n = 1 .$$

<u>ANOTHER</u> <u>EXAMPLE</u> : $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k^2+3k+2}$

AT FIRST THIS DOESN'T LOOK AT ALL RIGHT AND JUST WRITING OUT THE PARTIAL SUMS DOESN'T LEAD TO ANY TELESCOPING. HOWEVER,

$$\frac{1}{k^2+3k+2} = \frac{1}{(k+1)(k+2)} = \frac{A}{k+1} + \frac{B}{k+2}$$

(<u>PARTIAL</u> <u>FRACTIONS</u>)

$$1 = A(k+2) + B(k+1)$$

$$k = -1 : \quad 1 = A$$

$$k = -2 : \quad 1 = -B$$

$$\frac{1}{k^2+3k+2} = \frac{1}{k+1} - \frac{1}{k+2}$$

SO

$$\sum_{k=0}^{\infty} \frac{1}{k^2+3k+2} = \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k+2} \right)$$

NOW WRITE OUT THE PARTIAL SUMS :

$$S_0 = \frac{1}{1} - \frac{1}{2}$$

$$S_1 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right)$$

$$S_2 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$\vdots$$

$$S_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= 1 - \frac{1}{n+2}$$

THUS,

$$\sum_{k=0}^{\infty} \frac{1}{k^2+3k+2} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+2}\right) = 1.$$

THAT'S BASICALLY THE END OF THE EASY EXAMPLES. FOR A WHILE NOW WE WILL CONCENTRATE ON JUST DECIDING WHETHER OR NOT A SERIES CONVERGES AND NOT WORRY SO MUCH ABOUT WHAT IT CONVERGES TO, E.G., IT WILL SOON BE QUITE EASY TO SHOW THAT $\sum_{k=1}^{\infty} \frac{1}{k^2}$ CONVERGES, BUT IT TAKES A GREAT DEAL OF WORK TO FIND WHAT IT CONVERGES TO ($\frac{\pi^2}{6}$ ACTUALLY).