INTEGRAL THEOREMS :

THIS FINAL SECTION IS THE CULMINATION OF EVERYTHING WE HAVE DONE. IT REVOLVES AROUND TWO REMARKABLE THEOREMS OF CAUCHY AND A FEW (VERY FEW) OF THEIR APPLICATIONS. TO PROVIDE SOME PERSPECTIVE WE WILL STATE THESE THEOREMS AND A FEW CONSEQUENCES IMMEDIATELY.

SUPPOSE $f(z)$ IS ANALYTIC ON AND INSIDE THE SIMPLE CLOSED CONTOUR C. THEN

1. (CAUCHY - GOURSAT THEOREM)

$$\oint_C f(z)\,dz = 0$$

2. (CAUCHY INTEGRAL THEOREM) IF z_0 IS ANY POINT INSIDE C,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\,dz$$

WHEN C IS TRAVERSED COUNTERCLOCKWISE.

NOTE : THE CAUCHY INTEGRAL FORMULA SAYS THAT THE VALUES OF $f(z)$ INSIDE C ARE COMPLETELY DETERMINED BY ITS VALUES ON C.

A FEW CONSEQUENCES :

1. IF $f(z)$ IS ANALYTIC ON AND INSIDE THE SIMPLE CLOSED CONTOUR C AND z_0 IS INSIDE C THEN $f(z)$ HAS DERIVATIVES OF ALL ORDERS AT z_0 AND THESE ARE GIVEN BY

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

MOREOVER, THE <u>TAYLOR</u> <u>SERIES</u>

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$$

CONVERGES ON SOME DISC $|z-z_0| < R$ ABOUT z_0 TO $f(z)$.

<u>NOTE</u> : THE SITUATION IS WILDLY DIFFERENT FOR REAL FUNCTIONS $f(x)$. THESE CAN HAVE ONE DERIVATIVE AND YET FAIL TO HAVE EVEN A SECOND DERIVATIVE ; THEY CAN HAVE DERIVATIVES OF ALL ORDERS AND YET THE TAYOR SERIES CAN FAIL TO CONVERGE AND, EVEN IF IT DOES CONVERGE, IT MAY NOT CONVERGE TO $f(x)$.

2. IF $f(z)$ IS ANALYTIC ON THE ANNULUS $r < |z-z_0| < R$, THEN IT HAS A <u>LAURENT</u> <u>SERIES</u> EXPANSION

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

THERE AND

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

WHERE C IS ANY SIMPLE CLOSED CONTOUR IN THE ANNULUS WITH z_0 IN ITS INTERIOR (ORIENTED COUNTERCLOCKWISE).

<u>NOTE</u> : IN PARTICULAR,

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz$$

E.G., ON $0 < |z| < \infty$,

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

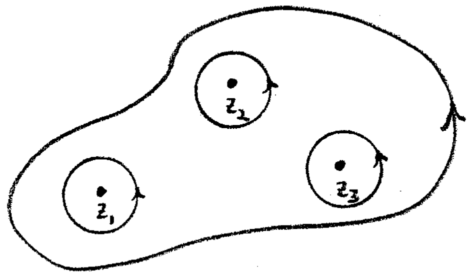
$$= \cdots + \frac{1}{2!} z^{-2} + \frac{1}{1!} z^{-1} + \frac{1}{0!}$$

SO $a_{-1} = 1$ AND

$$\oint_{|z|=1} e^{\frac{1}{z}}\, dz = 2\pi i\, a_{-1} = 2\pi i$$

3. (RESIDUE THEOREM) IF $f(z)$ IS ANALYTIC ON AND INSIDE A SIMPLE CLOSED CONTOUR C (ORIENTED COUNTERCLOCKWISE) EXCEPT FOR A FINITE NUMBER OF SINGULARITIES $z_1, \ldots, z_n$ INSIDE C, THEN

$$\oint_C f(z)\,dz = 2\pi i \left(\operatorname*{Res}_{z=z_1} f(z) + \cdots + \operatorname*{Res}_{z=z_n} f(z) \right)$$

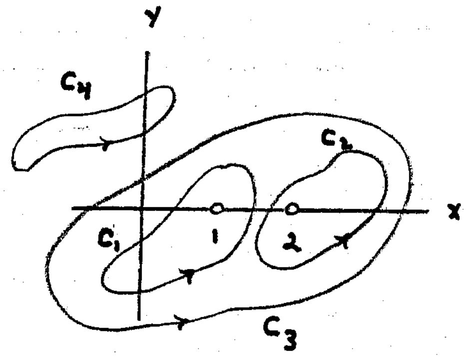


E.G., FROM THE EXAMPLE ON PAGES 9 - 10 OF "SERIES EXPANSIONS",

$$f(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{(z-1)(z-2)}$$

$$\operatorname*{Res}_{z=1} f(z) = -1 \quad \text{AND} \quad \operatorname*{Res}_{z=2} f(z) = 1$$

SO



$$\oint_{C_1} \frac{1}{z^2-3z+2}\,dz = 2\pi i \operatorname*{Res}_{z=1} f(z) = -2\pi i$$

$$\oint_{C_2} \frac{1}{z^2-3z+2}\,dz = 2\pi i \operatorname*{Res}_{z=2} f(z) = 2\pi i$$

$$\oint_{C_3} \frac{1}{z^2-3z+2}\,dz = 2\pi i \left(\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=2} f(z) \right) = 0$$

$$\oint_{C_4} \frac{1}{z^2-3z+2}\,dz = 0 \quad \text{(CAUCHY-GOURSAT THEOREM)}$$

CAUCHY'S PROOF OF THE CAUCHY-GOURSAT THEOREM IS SIMPLE, BUT IT USES AN ADDITIONAL HYPOTHESIS ($f'(z)$ CONTINUOUS). WE WILL SETTLE FOR THIS (GOURSAT'S PROOF DOES NOT USE THIS ADDITIONAL HYPOTHESIS, BUT IS SUBSTANTIALLY MORE COMPLICATED).

<u>GREEN'S THEOREM</u> : LET C BE A SIMPLE CLOSED CONTOUR, ORIENTED COUNTERCLOCKWISE, AND LET R BE THE REGION ON AND INSIDE C. IF $P(x,y)$ AND $Q(x,y)$ ARE REAL-VALUED FUNCTIONS WITH CONTINUOUS FIRST PARTIAL DERIVATIVES ON R, THEN

$$\oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA .$$

NOW SUPPOSE $f(z)$ IS ANALYTIC ON AND INSIDE C (AND $f'(z)$ IS CONTINUOUS). THEN IF $f(z) = u(x,y) + i\,v(x,y)$,

$$\oint_C f(z)\,dz = \oint_C u\,dx - v\,dy + i \oint_C v\,dx + u\,dy$$

(SEE PAGE 3 OF "CONTOUR INTEGRALS")

$$= \iint_R (-v_x - u_y)\,dA + i \iint_R (u_x - v_y)\,dA$$

(GREEN'S THEOREM)

$$= \iint_R 0\,dA + i \iint_R 0\,dA$$

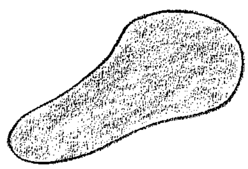
(CAUCHY-RIEMANN EQUATIONS)

$$= 0$$

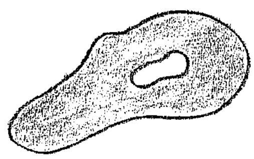
CAUCHY-GOURSAT THEOREM REPHRASED:

A DOMAIN D IS SAID TO BE <u>SIMPLY CONNECTED</u> IF EVERY SIMPLE CLOSED CONTOUR IN D ENCLOSES ONLY POINTS OF D.

E.G.,



BUT <u>NOT</u>

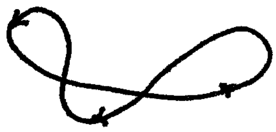


<u>CAUCHY-GOURSAT THEOREM</u>: IF $f(z)$ IS ANALYTIC ON A SIMPLY CONNECTED DOMAIN, THEN

$$\oint_C f(z)\,dz = 0$$

FOR EVERY SIMPLE CLOSED CONTOUR C IN D.

" SIMPLE " CAN BE DELETED :



CONSEQUENCE OF CAUCHY-GOURSAT :

$f(z)$ ANALYTIC ON A SIMPLY CONNECTED DOMAIN D, z_1 AND z_2 IN D, C_1 AND C_2 CONTOURS FROM z_1 TO z_2. THEN

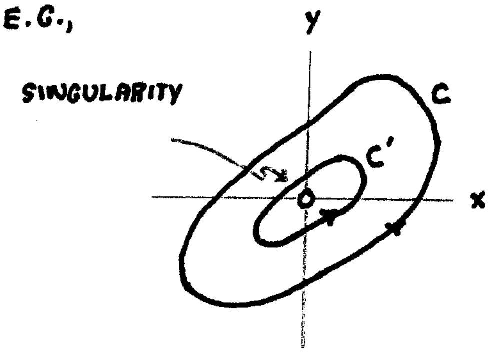
$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

(BECAUSE $\displaystyle\oint_{C_1+(-C_2)} f(z)dz = 0 = \int_{C_1} f(z)dz - \int_{C_2} f(z)dz$)

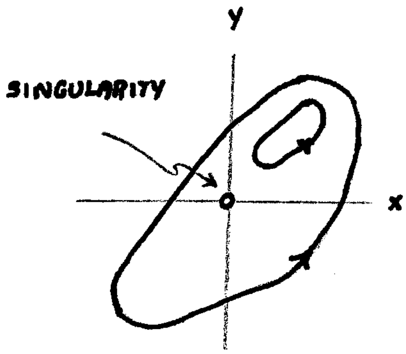
<u>CONTINUOUS</u> <u>DEFORMATIONS</u> :

LET $f(z)$ BE ANALYTIC ON SOME DOMAIN D. SUPPOSE C AND C' ARE TWO CURVES IN D AND THAT C CAN BE "CONTINUOUSLY DEFORMED" INTO C' WITHOUT PASSING THROUGH ANY SINGULARITIES OF $f(z)$.

E.G.,



THESE

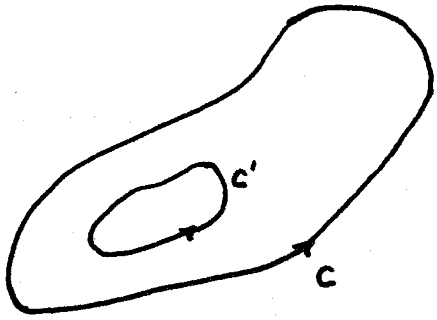


BUT NOT THESE

WE CLAIM THAT

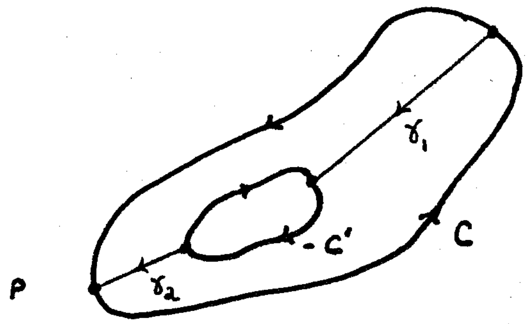
$$\int_{C'} f(z)dz = \int_{C} f(z)dz$$

TO SEE WHY :

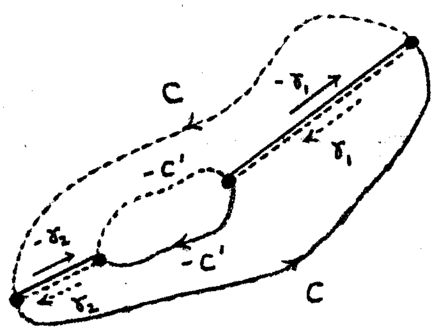


f(z) IS ANALYTIC EVERYWHERE BETWEEN C AND C'

REVERSE THE DIRECTION OF C' AND INSERT TWO EXTRA CURVES



NOW CONSIDER THE FOLLOWING CONTOUR :



f(z) IS ANALYTIC ON AND INSIDE IT SO THE INTEGRAL OVER IT IS ZERO.

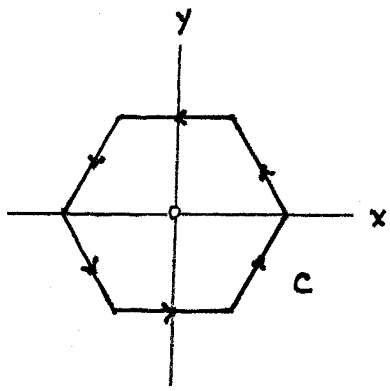
WRITING THE INTEGRAL AS THE SUM OVER THE EIGHT PIECES THE γs CANCEL AND WE GET

$$\int_C f(z)\,dz + \int_{-C'} f(z)\,dz = 0$$

$$\int_C f(z)\,dz - \int_{C'} f(z)\,dz = 0$$

AS REQUIRED.

E.G.,



$$\oint_C \frac{1}{z}\,dz = \oint_{|z|=1} \frac{1}{z}\,dz = 2\pi i$$ (PAGE 5 OF "CONTOUR INTEGRALS")

FOR A NUMBER OF TOPICS TO COME WE WILL NEED A TECHNICAL LEMMA ON THE "SIZE" OF A CONTOUR INTEGRAL.

<u>LEMMA</u>: LET $C : z(t)$, $a \le t \le b$, BE A CONTOUR OF LENGTH L AND LET $f(z)$ BE CONTINUOUS ON C AND SATISFY $|f(z)| \le M$ FOR EVERY z ON C. THEN

$$\left| \int_C f(z)\,dz \right| \le ML$$

WE WON'T PROVE THIS CAREFULLY BUT THE REASON IT IS TRUE IS CLEAR ENOUGH :

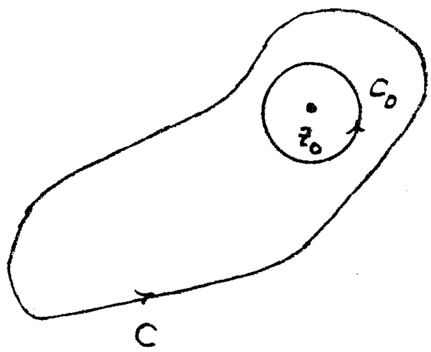
$$\left| \int_C f(z)\,dz \right| = \left| \int_a^b f(z(t))\,z'(t)\,dt \right|$$

$$\leq \int_a^b |f(z(t))|\,|z'(t)|\,dt \qquad \text{(THINK ABOUT THE RIEMANN SUMS)}$$

$$\leq M \int_a^b |z'(t)|\,dt = ML$$

WITH THIS WE ARE PREPARED TO SEE WHY THE CAUCHY INTEGRAL THEOREM IS TRUE.

$f(z)$ ANALYTIC ON AND INSIDE C AND z_0 INSIDE C



$$\oint_C \frac{f(z)}{z-z_0}\,dz = \oint_{C_0} \frac{f(z)}{z-z_0}\,dz$$ FOR <u>ANY</u> CIRCLE ABOUT z_0 CONTAINED INSIDE C

THUS, WE JUST NEED TO SHOW THAT

$$\oint_{C_0} \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0))$$

FOR THIS IT IS ENOUGH TO SHOW THAT

$$\left| \oint_{C_0} \frac{f(z)}{z-z_0}\,dz - 2\pi i\, f(z_0) \right| = 0$$

WE'LL DO THIS IN SORT OF AN ODD WAY. WE'LL SHOW THAT THE NUMBER ON THE LEFT-HAND SIDE HAS TO BE SMALLER THAN <u>ANY</u> POSITIVE NUMBER ε.

$$\oint_{C_0} \frac{f(z)}{z-z_0}\,dz = \oint_{C_0} \frac{f(z_0) + [f(z) - f(z_0)]}{z-z_0}\,dz$$

$$= f(z_0) \underbrace{\oint_{C_0} \frac{1}{z-z_0}\,dz}_{2\pi i \text{ (EXERCISE 66)}} + \oint_{C_0} \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

$$\oint_{C_0} \frac{f(z)}{z-z_0}\,dz - 2\pi i\, f(z_0) = \oint_{C_0} \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

$$\left| \oint_{C_0} \frac{f(z)}{z-z_0}\,dz - 2\pi i\, f(z_0) \right| = \left| \oint_{C_0} \frac{f(z)-f(z_0)}{z-z_0}\,dz \right|$$

$$\leq ML$$

WHERE L IS THE LENGTH OF C_0 AND $\left| \dfrac{f(z)-f(z_0)}{z-z_0} \right| \leq M$ ON C_0.

SINCE $f(z)$ IS CONTINUOUS AT z_0 WE CAN CHOOSE THE RADIUS r OF C_0 SMALL ENOUGH THAT $|f(z)-f(z_0)| < \frac{\varepsilon}{2\pi}$ ON C. THEN, ON C_0,

$$\left| \frac{f(z)-f(z_0)}{z-z_0} \right| = \frac{|f(z)-f(z_0)|}{|z-z_0|} < \frac{\varepsilon/2\pi}{r} = \frac{\varepsilon}{2\pi r}$$

SINCE $L = 2\pi r$,

$$\left| \oint_{C_0} \frac{f(z)}{z-z_0}\,dz - 2\pi i\, f(z_0) \right| < \frac{\varepsilon}{2\pi r}(2\pi r) = \varepsilon$$

THIS COMPLETES THE PROOF OF THE CAUCHY INTEGRAL THEOREM AND WE NOW USE IT TO EVALUATE SOME INTEGRALS.

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

WHENEVER $f(z)$ IS ANALYTIC ON AND INSIDE C, z_0 IS INSIDE C AND C IS ORIENTED COUNTERCLOCKWISE.

<u>EXAMPLES</u> : (ALL CURVES ARE ORIENTED COUNTERCLOCKWISE)

1. $$\oint_{|z|=2} \frac{e^z}{z-1}\,dz = 2\pi i e^1 = 2\pi i e$$

2. $$\oint_{|z|=\frac{1}{2}} \frac{e^z}{z-1}\,dz = 0 \quad \text{(CAUCHY-GOURSAT)}$$

FOR THE NEXT FOUR EXAMPLES WE COMPUTE

$$\oint_C \frac{\cos z}{z^3+z}\,dz$$

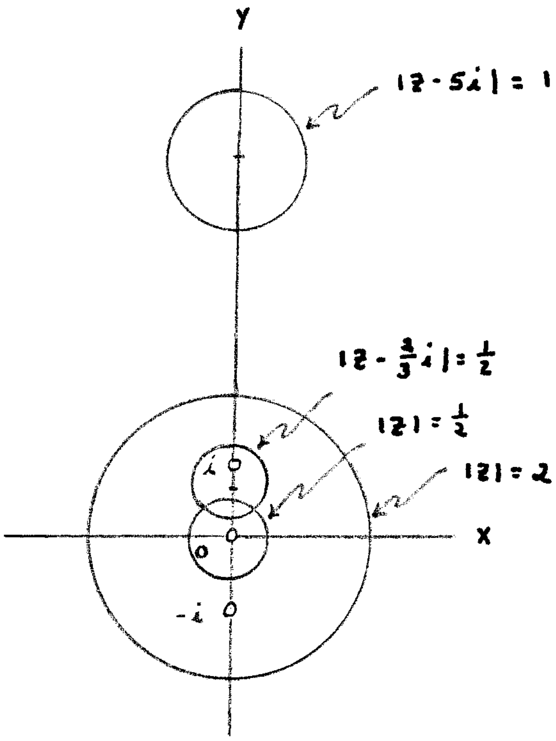
WHERE C IS

3. $|z| = \frac{1}{2}$

4. $|z - \frac{2}{3}i| = \frac{1}{2}$

5. $|z| = 2$

6. $|z-5i| = 1$



$$\frac{\cos z}{z^3+z} = \frac{\cos z}{z(z-i)(z+i)} \qquad \text{SINGULARITIES AT } -i,\ 0,\ i$$

3. $\dfrac{\cos z}{z^2+1}$ IS ANALYTIC ON AND INSIDE $|z| = \frac{1}{2}$ AND $z_0 = 0$ IS INSIDE $|z| = \frac{1}{2}$ SO

$$\oint_{|z|=\frac{1}{2}} \frac{\frac{\cos z}{z^2+1}}{z-0}\,dz = 2\pi i \left(\frac{\cos 0}{0^2+1}\right) = 2\pi i$$

4. $\dfrac{\cos z}{z(z+i)}$ IS ANALYTIC ON AND INSIDE $|z - \frac{2}{3}i| = \frac{1}{2}$ AND $z_0 = i$ IS INSIDE $|z - \frac{2}{3}i| = \frac{1}{2}$ SO

$$\oint_{|z-\frac{2}{3}i|=\frac{1}{2}} \frac{\frac{\cos z}{z(z+i)}}{z-i} = 2\pi i \left(\frac{\cos i}{i(i+i)}\right) = (-\pi\cosh 1)i$$

5. $|z| = 2$ CONTAINS ALL THREE SINGULARITIES SO THE SAME TRICK WON'T WORK. HOWEVER,

$$\frac{1}{z(z-i)(z+i)} = \frac{1}{z} - \frac{1}{2}\,\frac{1}{z-i} - \frac{1}{2}\,\frac{1}{z+i} \qquad \text{(PARTIAL FRACTIONS)}$$

SO

$$\oint_{|z|=2} \frac{\cos z}{z^3+z}\,dz = \oint_{|z|=2} \frac{\cos z}{z-0}\,dz - \frac{1}{2}\oint_{|z|=2} \frac{\cos z}{z-i}\,dz - \frac{1}{2}\oint_{|z|=2} \frac{\cos z}{z-(-i)}\,dz$$

$$= 2\pi i\left[\cos 0 - \frac{1}{2}\cos(i) - \frac{1}{2}\cos(-i)\right]$$

$$= 2\pi i\left[1 - \cosh 1\right]$$

6. $$\oint_{|z-5i|=1} \frac{\cos z}{z^3+z}\,dz = 0 \qquad \text{(CAUCHY-GOURSAT)}$$

OUR FINAL EXAMPLE MAY LOOK A BIT WEIRD, BUT IT HAS A VERY INTERESTING APPLICATION.

7. $$\oint_{|z|=1} \frac{1}{z^2+4iz-1}\,dz$$, WITH $|z|=1$ ORIENTED COUNTERCLOCKWISE

SINGULARITIES : $z^2+4iz-1=0 \Rightarrow z = (-2\pm\sqrt{3})i$

(QUADRATIC FORMULA)

$$z^2+4iz-1 = \left(z-(\sqrt{3}-2)i\right)\left(z+(\sqrt{3}+2)i\right)$$

ONLY $(\sqrt{3}-2)i$ IS INSIDE $|z|=1$ SO

$$\oint_{|z|=1} \frac{1}{z^2+4iz-1}\,dz = \oint_{|z|=1} \frac{\frac{1}{z+(\sqrt{3}+2)i}}{z-(\sqrt{3}-2)i}\,dz$$

$$= 2\pi i\left(\frac{1}{(\sqrt{3}-2)i+(\sqrt{3}+2)i}\right) = \frac{2\pi i}{2\sqrt{3}\,i}$$

$$= \frac{\pi}{\sqrt{3}}$$

NOW FOR THE APPLICATION.

CONTOUR INTEGRALS CAN OFTEN BE USED TO EVALUATE OTHERWISE INTRACTIBLE DEFINITE INTEGRALS, E.G.,

$$\int_0^{2\pi} \frac{1}{2+\sin\theta}\,d\theta$$

(TRY YOUR FAVORITE INTEGRATION TECHNIQUE ON THIS - IT WON'T WORK!)

THE IDEA IS TO TRY TO REGARD THIS AS THE RESULT OF PARAMETRIZING SOME CONTOUR INTEGRAL $\int_{|z|=1} f(z)dz = \int_0^{2\pi} f(e^{i\theta})ie^{i\theta}d\theta$ AND DO THE CONTOUR INTEGRAL INSTEAD. THUS, WE WANT TO FIND AN $f(z)$ FOR WHICH

$$\int_0^{2\pi} f(e^{i\theta})ie^{i\theta}d\theta = \int_0^{2\pi} \frac{1}{2+\sin\theta}\,d\theta$$

$$= \int_0^{2\pi} \frac{-ie^{-i\theta}}{2+\sin\theta}\,ie^{i\theta}d\theta$$

THUS, $f(z)$ SHOULD SATISFY

$$f(e^{i\theta}) = \frac{-ie^{-i\theta}}{2+\sin\theta}$$

BUT

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

$$= \frac{1}{2i}(z - \tfrac{1}{z}) \qquad \text{ON } |z| = 1$$

SO

$$\frac{-ie^{-i\theta}}{2+\sin\theta} = \frac{-i(\frac{1}{z})}{2+\frac{1}{2i}(z-\frac{1}{z})} \quad \frac{2iz}{2iz}$$

$$= \frac{2}{4iz+z^2-1}$$

$$= 2\left(\frac{1}{z^2+4iz-1}\right).$$

THUS,

$$\int_0^{2\pi} \frac{1}{2+\sin\theta}\,d\theta = \int_0^{2\pi} \frac{-ie^{-i\theta}}{2+\sin\theta}(ie^{i\theta})\,d\theta$$

$$= 2\oint_{|z|=1} \frac{1}{z^2+4iz-1}\,dz$$

$$= \frac{2\pi}{\sqrt{3}} \qquad \text{BY EXAMPLE 6}$$

MANY MORE EXAMPLES OF THIS SORT CAN BE GOTTEN FROM THE RESIDUE THEOREM.

EXERCISES: (ALL CURVES ARE ORIENTED COUNTERCLOCKWISE)

70. FIND THE VALUE OF $\displaystyle\oint_{|z|=1} \log(z+2)\,dz$. ANS. 0

71. FIND THE VALUE OF $\displaystyle\oint_C \frac{z}{2z+1}\,dz$, WHERE C IS THE BOUNDARY OF THE SQUARE WHOSE SIDES LIE ALONG THE LINES $x = \pm 2$ AND $y = \pm 2$.

ANS. $-\frac{\pi}{2} i$

72. FIND THE VALUE OF $\displaystyle\oint_{|z-i|=2} \frac{1}{z^2+4}\,dz$

ANS. $\frac{\pi}{2}$

73. FIND THE VALUE OF $\displaystyle\oint_{|z|=3} \frac{\sin(\pi z^2) + \cos(\pi z^2)}{z^2 - 3z + 2}\,dz$

ANS. $4\pi i$

74. SUPPOSE $f(z)$ AND $g(z)$ ARE ANALYTIC ON AND INSIDE THE SIMPLE CLOSED CONTOUR C AND $f(z) = g(z)$ FOR ALL z ON C. SHOW THAT $f(z) = g(z)$ FOR ALL z INSIDE C AS WELL.

75. FIND THE VALUE OF $\displaystyle\oint_{|z-z_0|=1} \frac{z^2+1}{z^2-1}\,dz$ IF

(a) $z_0 = 1$ ANS. $2\pi i$

(b) $z_0 = \frac{1}{2}$ ANS. $2\pi i$

(c) $z_0 = -1$ ANS. $-2\pi i$

(d) $z_0 = i$ ANS. 0

76. FIND THE VALUE OF $\int_0^{2\pi} \frac{1}{1-2a\cos\theta+a^2}\,d\theta$, WHERE $0<a<1$.

ANS. $\frac{2\pi}{1-a^2}$

NOW LET'S LOOK AGAIN AT THE CAUCHY INTEGRAL FORMULA :

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz$$

IF Δz IS SUCH THAT $z_0+\Delta z$ IS ALSO INSIDE C, THEN

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{1}{\Delta z}\left[\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-(z_0+\Delta z)}\,dz - \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz\right]$$

$$= \frac{1}{\Delta z}\,\frac{1}{2\pi i}\oint_C \left[\frac{1}{z-(z_0+\Delta z)} - \frac{1}{z-z_0}\right] f(z)\,dz$$

$$= \frac{1}{\Delta z}\,\frac{1}{2\pi i}\oint_C \frac{\Delta z}{(z-(z_0+\Delta z))(z-z_0)}\,f(z)\,dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-(z_0+\Delta z))(z-z_0)}\,dz$$

SO

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz$$

SIMILARLY, FOR $n \geq 0$,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

IN PARTICULAR,

$f(z)$ HAS DERIVATIVES OF ALL ORDERS EVERYWHERE INSIDE C .

ONE CAN THEREFORE AT LEAST BUILD THE TAYLOR SERIES

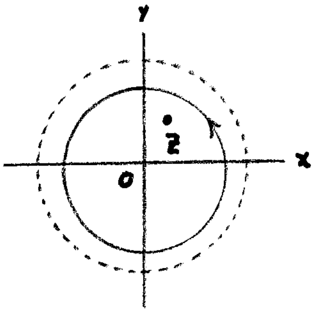
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

AT EACH z_0 INSIDE C.

REMARKABLY, THE SERIES CAN ACTUALLY BE SHOWN TO CONVERGE ON SOME DISC ABOUT z_0 AND, MOREOVER, TO CONVERGE TO $f(z)$ THERE (COMPARE THIS WITH THE FIRST <u>NOTE</u> ON PAGE 2).

> <u>NOTE</u> : WE WILL NOT GO THROUGH THE ANALYSIS TO PROVE THIS, BUT IT IS WORTH A MOMENT TO SEE WHY THE SITUATION IS SO DIFFERENT FOR COMPLEX FUNCTIONS. THE KEY IS THE CAUCHY INTEGRAL FORMULAS. TO EASE THE NOTATION WE'LL TAKE $z_0 = 0$.

$f(z)$ ANALYTIC ON $|z| < R$. FIX SUCH A z AND LET C BE A CIRCLE OF RADIUS $r < R$ INSIDE AND WITH z IN ITS INTERIOR. THEN



$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(s)}{s-z}\,ds$$

USE $\quad 1 + a + a^2 + \cdots + a^{N-1} = \dfrac{1-a^N}{1-a} = \dfrac{1}{1-a} - \dfrac{a^N}{1-a}$ TO WRITE

$$\frac{1}{s-z} = \frac{1}{s}\left(\frac{1}{1-\frac{z}{s}}\right) = \frac{1}{s} + \frac{1}{s^2}z + \frac{1}{s^3}z^2 + \cdots + \frac{1}{s^N}z^{N-1} + \frac{1}{(s-z)s^N}z^N$$

SO THAT

$$\frac{f(s)}{s-z} = \frac{f(s)}{s-0} + \frac{f(z)}{(s-0)^2}z + \frac{f(s)}{(s-0)^3}z^2 + \cdots + \frac{f(s)}{(s-0)^N}z^{N-1} + \frac{f(s)}{(s-z)s^N}z^N$$

INTEGRATE OVER C TO GET

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(s)}{s-z}ds = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \cdots + \frac{f^{(N-1)}(0)}{(N-1)!}z^{N-1} + \left(\frac{1}{2\pi i}\oint_C \frac{f(s)}{(s-z)s^N}ds\right)z^N$$

NOW ONE USES THE LEMMA ON PAGE 8 TO BOUND

$$\left|\left(\frac{1}{2\pi i}\oint_C \frac{f(s)}{(s-z)s^N}ds\right)z^N\right|$$

AND SHOW THAT IT $\to 0$ AS $N \to \infty$ AND THIS PROVES THE RESULT.

ANALOGOUS ARGUMENTS PROVE

<u>LAURENT'S</u> <u>THEOREM</u> : IF $f(z)$ IS ANALYTIC ON THE ANNULUS

$$r < |z-z_0| < R$$

THEN

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

WHERE

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(s)}{(s-z)^{n+1}}ds$$

AND C IS ANY CLOSED CONTOUR (ORIENTED COUNTERCLOCKWISE) IN THE ANNULUS SURROUNDING z_0 .

RECALL (PAGE 14 OF "SERIES EXPANSIONS") THAT IN THE SPECIAL CASE OF A LAURENT EXPANSION ON

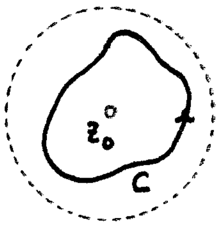
$$0 < |z - z_0| < R$$

THE COEFFICIENT

$$a_{-1} = \operatorname*{Res}_{z=z_0} f(z)$$

IS CALLED THE RESIDUE OF $f(z)$ AT z_0. IT'S SIGNIFICANCE IS THAT

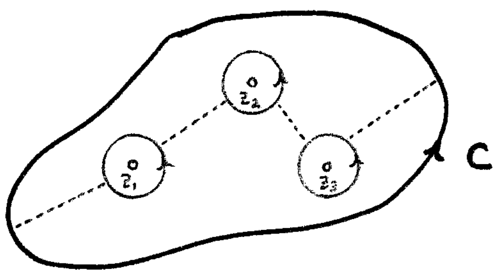
$$\oint_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_0} f(z)$$



MORE GENERALLY,

<u>RESIDUE</u> <u>THEOREM</u> : LET C BE A SIMPLE CLOSED CONTOUR AND SUPPOSE $f(z)$ IS ANALYTIC ON AND INSIDE C EXCEPT FOR A FINITE NUMBER OF SINGULARITIES $z_1, \ldots, z_n$ INSIDE C. THEN

$$\oint_C f(z)\,dz = 2\pi i \left[\operatorname*{Res}_{z=z_1} f(z) + \cdots + \operatorname*{Res}_{z=z_n} f(z) \right]$$



PICTURE OF THE PROOF

TO USE THE RESIDUE THEOREM EFFICIENTLY WE NEED A QUICKER WAY OF COMPUTING RESIDUES OF CERTAIN FUNCTIONS.

<u>LEMMA</u>: SUPPOSE $f(z) = \dfrac{p(z)}{q(z)}$, WHERE p AND q ARE BOTH ANALYTIC AT z_0, $q(z_0) = 0$, $p(z_0) \neq 0$ AND $q'(z_0) \neq 0$. THEN $f(z)$ HAS A SIMPLE POLE AT z_0 AND

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)}$$

<u>PROOF</u>:

$$f(z) = \frac{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots}{b_1(z-z_0) + b_2(z-z_0)^2 + \cdots}$$

$$(b_0 = 0,\ a_0 \neq 0,\ b_1 \neq 0)$$

$$= \frac{\dfrac{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots}{b_1 + b_2(z-z_0) + b_3(z-z_0)^2 + \cdots}}{z-z_0}$$

$$= \frac{\text{ANALYTIC AT } z_0 \text{ WITH NONZERO CONSTANT TERM } \frac{a_0}{b_1}}{z-z_0}$$

SO

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{a_0}{b_1} = \frac{p(z_0)}{q'(z_0)} \qquad \square$$

E.G. $f(z) = \dfrac{\cos z}{e^z - 1}$ AT $z_0 = 0$

$$\operatorname*{Res}_{z=0} \frac{\cos z}{e^z - 1} = \frac{\cos 0}{e^0} = 1$$

A FEW INTEGRALS : (ALL CURVES ARE ORIENTED COUNTERCLOCKWISE)

1. $$\oint_{|z|=\frac{1}{3}} \frac{\cos z}{z^2(z-1)}\,dz$$

SINGULARITIES : $z=0,1$

ONLY $z=0$ IS INSIDE $|z|=\frac{1}{3}$

(POLE OF ORDER 2)

$$= \oint_{|z|=\frac{1}{3}} \frac{\cos z/z-1}{(z-0)^2}\,dz$$

USE $f^{(n)}(z_0) = \dfrac{n!}{2\pi i}\displaystyle\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$

WITH $n=1$

$$= \frac{2\pi i}{1!}\left(\frac{\cos z}{z-1}\right)'_{z=0} = 2\pi i\left[\frac{(z-1)\sin z - \cos z}{(z-1)^2}\right]_{z=0}$$

$$= 2\pi i\left[\frac{0-1}{1}\right] = -2\pi i$$

2. $$\oint_{|z-1|=3} \frac{15z+9}{z^3-9z}\,dz$$

SINGULARITIES : $z = 0, 3, -3$

ONLY $z=0, 3$ ARE INSIDE $|z-1|=3$ AND BOTH ARE SIMPLE POLES.

$p(z) = 15z+9$	$p(0)=9$	$p(3) = 54$
$q(z) = z^3-9z$	$q(0)=0$	$q(3)=0$
$q'(z) = 3z^2-9$	$q'(0)=-9$	$q'(3)=18$

$$\operatorname*{Res}_{z=0} \frac{15z+9}{z^3-9z} = \frac{9}{-9} = -1$$

$$\operatorname*{Res}_{z=3} \frac{15z+9}{z^3-9z} = \frac{54}{18} = 3$$

$$= 2\pi i\left[-1+3\right]$$

$$= 4\pi i$$

3. $\displaystyle\oint_{|z|=1} \frac{e^{-z^2}}{\sin 2z}\, dz$

SINGULARITIES : $z = \frac{n\pi}{2}$, $n = 0, \pm 1, \ldots$

ONLY $z = 0$ IS INSIDE $|z| = 1$

AND IT IS A SIMPLE POLE.

$p(z) = e^{-z^2}$ $\quad p(0) = 1$

$q(z) = \sin 2z$ $\quad q(0) = 0$

$q'(z) = 2\cos 2z$ $\quad q'(0) = 2$

$$\operatorname*{Res}_{z=0} \frac{e^{-z^2}}{\sin 2z} = \frac{1}{2}$$

$= 2\pi i \left[\frac{1}{2}\right] = \pi i$

EXERCISES : COMPUTE THE FOLLOWING INTEGRALS (ALL CURVES ARE ORIENTED COUNTERCLOCKWISE)

77. $\displaystyle\oint_{|z|=2} \frac{\cosh z}{z^4}\, dz$ ANS. 0

78. $\displaystyle\oint_{|z-i|=2} \frac{1}{(z^2+4)^2}\, dz$ ANS. $\frac{\pi}{16}$

79. $\displaystyle\oint_{|z|=4} \frac{3z^3+2}{(z-1)(z^2+9)}\, dz$ ANS. $6\pi i$

80. $\displaystyle\oint_{|z|=2} \tan z\, dz$ ANS. $-4\pi i$

81. $\displaystyle\oint_{|z|=2} \frac{1}{\sinh 2z}\, dz$ ANS. $-\pi i$

THE RESIDUE THEOREM CAN BE USED TO EVALUATE A WIDE VARIETY OF ORDINARY DEFINITE INTEGRALS THAT CANNOT BE TREATED BY THE METHODS OF CALCULUS. WE WILL CONCLUDE BY DESCRIBING JUST ONE TYPE.

LET
$$f(x) = \frac{p(x)}{q(x)}$$

WHERE $p(x)$ AND $q(x)$ ARE POLYNOMIALS WITH NO COMMON FACTORS, $q(x)$ HAS NO REAL ZEROS AND $\deg q(x) \geq \deg p(x) + 2$, E.G.,
$$f(x) = \frac{1}{x^4+1} .$$

WE CONSIDER
$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

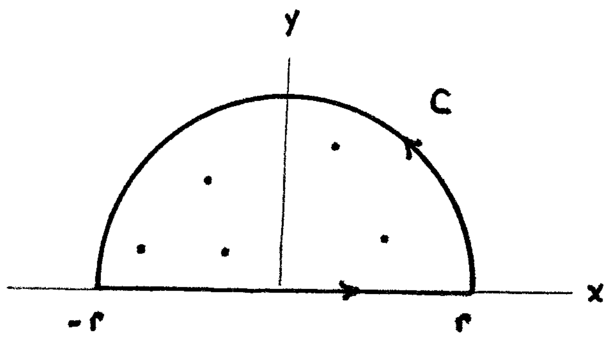
(THIS IS CALLED THE <u>PRINCIPAL</u> <u>VALUE</u> OF THE IMPROPER INTEGRAL).

PROCEDURE :

CONSIDER THE CORRESPONDING COMPLEX FUNCTION
$$f(z) = \frac{p(z)}{q(z)}$$

E.G., $\frac{1}{z^4+1}$. IT HAS FINITELY MANY SINGULARITIES, NONE ON THE REAL AXIS. WE CONSIDER ONLY THOSE IN THE UPPER HALF-PLANE (IGNORE THE REST).

CHOOSE r LARGE ENOUGH THAT ALL OF SINGULARITIES OF $f(z)$ IN THE UPPER HALF PLANE ARE CONTAINED INSIDE THE SEMI-CIRCULAR CONTOUR



TWO EXPRESSIONS FOR $\oint_C f(z)dz$:

$$\oint_C f(z)dz = 2\pi i \sum_{Im>0} Res\, f(z)$$

$$= \int_{-r}^{r} f(x)dx + \int_{C_r} f(z)dz$$

SO

$$\int_{-r}^{r} f(x)dx = 2\pi i \sum_{Im>0} Res\, f(z) - \int_{C_r} f(z)dz$$

WHERE

$$C_r : z(t) = re^{it} , \quad 0 \le t \le \pi .$$

SINCE ALL OF THE SINGULARITIES WITH $Im > 0$ ARE INSIDE C_r THE SUM DOES NOT CHANGE AS $r \to \infty$ SO

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{Im>0} Res\, f(z) - \lim_{r\to\infty} \int_{C_r} f(z)dz .$$

IF ONE CAN SHOW THAT $\lim_{r\to\infty} \int_{C_r} f(z)dz = 0$, THEN

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{Im>0} Res\, f(z)$$

WE ILLUSTRATE WITH AN

<u>EXAMPLE</u> : $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$

$f(z) = \dfrac{1}{z^4+1}$ SINGULARITIES: $z^4+1 = 0 \Rightarrow$

$$z^4 = -1 = e^{\pi i} = e^{(\pi + 2k\pi)i}$$

SO

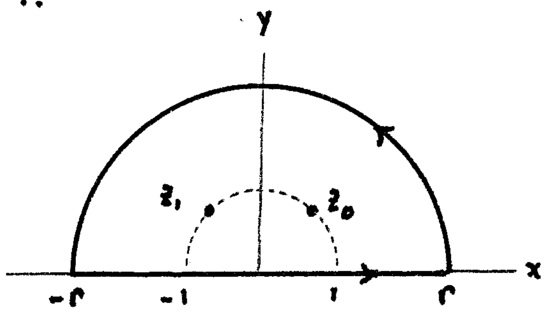
$$z = e^{\frac{\pi+2k\pi}{4}i}, \quad k = 0, 1, 2, 3$$

$$z_0 = e^{\frac{\pi}{4}i}, \quad z_1 = e^{\frac{3\pi}{4}i}, \quad z_2 = e^{\frac{5\pi}{4}i}, \quad z_3 = e^{\frac{7\pi}{4}i}$$

ALL ARE ON $|z| = 1$.

ONLY z_0 AND z_1 ARE IN $\text{Im} > 0$.

CHOOSE ANY $r > 1$.



$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = 2\pi i \left[\operatorname*{Res}_{z=z_0} f(z) + \operatorname*{Res}_{z=z_1} f(z) \right] - \lim_{r\to\infty} \int_{C_r} \frac{1}{z^4+1}\,dz$$

$$\operatorname*{Res}_{z=z_i} f(z) = \frac{p(z_i)}{q'(z_i)} = \frac{1}{4 z_i^3}$$

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{4} e^{-\frac{3\pi}{4}i} = -\frac{1}{4} e^{\frac{\pi}{4}i}$$

$$\operatorname*{Res}_{z=z_1} f(z) = \frac{1}{4} e^{-\frac{9\pi}{4}i} = \frac{1}{4} e^{-\frac{\pi}{4}i}$$

$$2\pi i\left[\operatorname*{Res}_{z=z_0} f(z) + \operatorname*{Res}_{z=z_1} f(z)\right] = 2\pi i\left[-\tfrac{1}{4}e^{\frac{\pi}{4}i} + \tfrac{1}{4}e^{-\frac{\pi}{4}i}\right]$$

$$= 2\pi i\left[-\frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i + \frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i\right]$$

$$= \frac{\pi}{\sqrt{2}}$$

SO

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = \frac{\pi}{\sqrt{2}} - \lim_{r\to\infty} \int_{C_r} \frac{1}{z^4+1}\,dz$$

NOW WE SHOW THAT THE LIMIT IS ZERO :

<u>NOTE</u> : WE NEED THE FOLLOWING CONSEQUENCE OF THE TRIANGLE INEQUALITY :

$$|z_1 + z_2| \geq |z_1| - |z_2|$$

PROOF :

$$|z_1| = |(z_1+z_2) - z_2| \leq |z_1+z_2| + |-z_2| = |z_1+z_2| + |z_2|$$

SO

$$|z_1| - |z_2| \leq |z_1+z_2| .$$

NOW, ON C_r ,

$$\left|\frac{1}{z^4+1}\right| = \frac{1}{|z^4+1|} \leq \frac{1}{|z|^4-1} = \frac{1}{r^4-1}$$

SINCE THE LENGTH OF C_r IS πr,

$$\left|\int_{C_r} \frac{1}{z^4+1}\,dz\right| \leq \frac{1}{r^4-1}(\pi r) = \frac{\pi r}{r^4-1} \to 0 \text{ AS } r\to\infty$$

THUS,

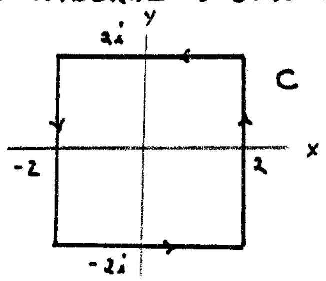
$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = \frac{\pi}{\sqrt{2}}$$

EXERCISE 82 : SHOW THAT $\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\,dx = \frac{\pi}{3}$

SOLUTIONS TO THE EXERCISES :

70. $\oint_{|z|=1} \log(z+2)\,dz$: $\log(z+2) = \log((x+2)+yi)$ IS ANALYTIC EXCEPT WHERE $y=0$ AND $x \le -2$ AND NONE OF THESE POINTS IS ON OR INSIDE $|z|=1$ SO THE INTEGRAL IS ZERO BY CAUCHY - GOURSAT.

71. $\oint_C \frac{z}{2z+1}\,dz$, WHERE



$$= \frac{1}{2}\oint_C \frac{z}{z-(-\frac{1}{2})}\,dz = \frac{1}{2}(2\pi i)(-\tfrac{1}{2}) = -\frac{\pi}{2}i$$

72. $$\oint_{|z-i|=2} \frac{1}{z^2+4}\,dz = \oint_{|z-i|=2} \frac{\frac{1}{z+2i}}{z-2i}\,dz = 2\pi i\left(\frac{1}{2i+2i}\right) = \frac{\pi}{2}$$

73. $$\oint_{|z|=3} \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)}\,dz =$$

$$\oint_{|z|=3} \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-2}\,dz - \oint_{|z|=3} \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-1}\,dz$$

(PARTIAL FRACTIONS)

73. (CONTINUED) SINCE $|z| = 3$ CONTAINS BOTH $z=1$ AND $z=2$, THE CAUCHY INTEGRAL FORMULA GIVES

$$2\pi i\,(\sin(\pi\cdot 2^2) + \cos(\pi\cdot 2^2)) - 2\pi i\,(\sin(\pi\cdot 1^2) + \cos(\pi\cdot 1^2))$$

$$= 2\pi i - (-2\pi i) = 4\pi i$$

74. FOR ANY z_0 INSIDE C,

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\oint_C \frac{g(z)}{z-z_0}\,dz \quad \text{SINCE } f(z) = g(z) \text{ ON } C$$

$$= g(z_0)$$

75. $$\oint_{|z-z_0|=1} \frac{z^2+1}{z^2-1}\,dz = \oint_{|z-z_0|=1} \left(1 + \frac{2}{z^2-1}\right)dz \quad \text{(BY LONG DIVISION)}$$

$$= \oint_{|z-z_0|=1} \frac{2}{z^2-1}\,dz \quad \text{(BY CAUCHY-GOURSAT)}$$

$$= \oint_{|z-z_0|=1} \frac{1}{z-1}\,dz - \oint_{|z-z_0|=1} \frac{1}{z+1}\,dz \quad \text{(BY PARTIAL FRACTIONS)}$$

(a) $z_0 = 1$: $|z-1| = 1$ CONTAINS $z = 1$, BUT NOT $z = -1$ SO

$$2\pi i - 0 = 2\pi i$$

(b) $z_0 = \frac{1}{2}$: $|z - \frac{1}{2}| = 1$ CONTAINS $z=1$, BUT NOT $z = -1$ SO

$$2\pi i - 0 = 2\pi i$$

(c) $z_0 = -1$: $|z+1| = 1$ CONTAINS $z = -1$, BUT NOT $z = 1$ SO

$$0 - 2\pi i = -2\pi i$$

(d) $z_0 = i$: $|z - i| = 1$ CONTAINS NEITHER SO

$$0 - 0 = 0$$

76. $$\int_0^{2\pi} \frac{1}{1-2a\cos\theta + a^2}\, d\theta \qquad (0 < a < 1)$$

$$= \int_0^{2\pi} \frac{-ie^{-i\theta}}{1-2a\cos\theta+a^2}\, ie^{i\theta} d\theta$$

$$\frac{-ie^{-i\theta}}{1-2a\cos\theta+a^2} = \frac{-\frac{i}{z}}{1-a(z+\frac{1}{z})+a^2} \quad \frac{z}{z}$$

$$= \frac{-i}{z-az^2-a+a^2z} = \frac{-i}{-az^2+(1+a^2)z-a}$$

$$= \frac{\frac{i}{a}}{z^2-\left(\frac{1+a^2}{a}\right)z+1} = \frac{\frac{i}{a}}{(z-a)(z-\frac{1}{a})}$$

$0 < a < 1 \Rightarrow$ ONLY $z = a$ IS INSIDE $|z| = 1$ SO

$$\int_0^{2\pi} \frac{1}{1-2a\cos\theta+a^2}\, d\theta = \oint_{|z|=1} \frac{\frac{i/a}{z-\frac{1}{a}}}{z-a}\, dz$$

$$= 2\pi i \left(\frac{\frac{i}{a}}{a-\frac{1}{a}} \right) = -2\pi \left(\frac{1}{a^2-1} \right)$$

$$= \frac{2\pi}{1-a^2}$$

77. $$\oint_{|z|=2} \frac{\cosh z}{z^4}\, dz = \oint_{|z|=2} \frac{\cosh z}{(z-0)^4}\, dz = \frac{2\pi i}{3!} (\cosh z)'''(0)$$

$$= \frac{2\pi i}{3!} \sinh 0 = 0$$

78. $$\oint_{|z-i|=2} \frac{1}{(z^2+4)^2}\,dz = \oint_{|z-i|=2} \frac{\left(\frac{1}{z+2i}\right)^2}{(z-2i)^2}\,dz = \frac{2\pi i}{1!}\left((z+2i)^{-2}\right)'(2i)$$

$$= 2\pi i\left(-\frac{2}{(2i+2i)^3}\right) = \frac{-4\pi i}{4^3 i^3} = \frac{\pi}{16}$$

79. $$\oint_{|z|=4} \frac{3z^3+2}{(z-1)(z^2+9)}\,dz$$ SINGULARITIES : $z = 1, -3i, 3i$

ALL THREE ARE INSIDE $|z|=4$

$$= 2\pi i\left[\operatorname*{Res}_{z=1} \frac{3z^3+2}{(z-1)(z^2+9)} + \operatorname*{Res}_{z=-3i} \frac{3z^3+2}{(z-1)(z^2+9)} + \operatorname*{Res}_{z=3i} \frac{3z^3+2}{(z-1)(z^2+9)}\right]$$

$p(z) = 3z^3+2$ $\quad q(z) = (z-1)(z^2+9)$ $\quad q'(z) = (z^2+9) + 2z(z-1)$

$p(1) = 5$ $\quad q(1) = 0$ $\quad q'(1) = 10$

$$\operatorname*{Res}_{z=1} \frac{3z^3+2}{(z-1)(z^2+9)} = \frac{5}{10} = \frac{1}{2}$$

$p(-3i) = 2+81i$ $\quad q(-3i) = 0$ $\quad q'(-3i) = -18+6i$

$$\operatorname*{Res}_{z=-3i} \frac{3z^3+2}{(z-1)(z^2+9)} = \frac{2+81i}{-18+6i} = \frac{15}{12} - \frac{49}{12}i$$

$p(3i) = 2-81i$ $\quad q(3i) = 0$ $\quad q'(3i) = -18-6i$

$$\operatorname*{Res}_{z=3i} \frac{3z^3+2}{(z-1)(z^2+4)} = \frac{2-81i}{-18-6i} = \frac{15}{12} + \frac{49}{12}i$$

THUS,

$$\oint_{|z|=4} \frac{3z^3+2}{(z-1)(z^2+9)}\,dz = 2\pi i\left[\frac{1}{2} + \frac{15}{12} - \frac{49}{12}i + \frac{15}{12} + \frac{49}{12}i\right]$$

$$= 2\pi i\,[3]$$

$$= 6\pi i$$

80. $\displaystyle\oint_{|z|=2} \tan z\, dz = \oint_{|z|=2} \frac{\sin z}{\cos z}\, dz$: SINGULARITIES: $z = (n + \frac{1}{2})\pi$, $n = 0, \pm 1, \cdots$

ONLY $\frac{\pi}{2}$ AND $-\frac{\pi}{2}$ ARE INSIDE $|z| = 2$

$$\operatorname*{Res}_{z=\frac{\pi}{2}} \frac{\sin z}{\cos z} = \frac{\sin \frac{\pi}{2}}{-\sin \frac{\pi}{2}} = -1$$

$$= \operatorname*{Res}_{z=-\frac{\pi}{2}} \frac{\sin z}{\cos z}$$

$$\oint_{|z|=2} \tan z\, dz = 2\pi i\, [-1-1] = -4\pi i$$

81. $\displaystyle\oint_{|z|=2} \frac{1}{\sinh 2z}\, dz$: SINGULARITIES: $2z = n\pi i$, $n = 0, \pm 1, \cdots$

$z = \frac{n\pi}{2} i$

INSIDE $|z| = 2$ ONLY FOR

$n = -1, 0, 1$

$$\operatorname*{Res}_{z=0} \frac{1}{\sinh 2z} = \frac{1}{2\cosh 2\cdot 0} = \frac{1}{2}$$

$$\operatorname*{Res}_{z=\frac{\pi}{2}i} \frac{1}{\sinh 2z} = \frac{1}{2\cosh 2(\frac{\pi}{2}i)} = \frac{1}{2\cosh(\pi i)}$$

$$= \frac{1}{e^{\pi i} + e^{-\pi i}} = \frac{1}{-2} = -\frac{1}{2}$$

$$= \operatorname*{Res}_{z=-\frac{\pi}{2}i} \frac{1}{\sinh 2z}$$

THUS,

$$\oint_{|z|=2} \frac{1}{\sinh 2z}\, dz = 2\pi i \left[\frac{1}{2} - \frac{1}{2} - \frac{1}{2} \right] = -\pi i$$

82. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\,dx$

$f(z) = \dfrac{z^2}{z^6+1}$ SINGULARITIES : $z^6+1=0 \Rightarrow z^6 = -1 = e^{(2k+1)\pi i}$

$\Rightarrow z = e^{(2k+1)\pi i/6}$, $k = 0, 1, \dots, 5$

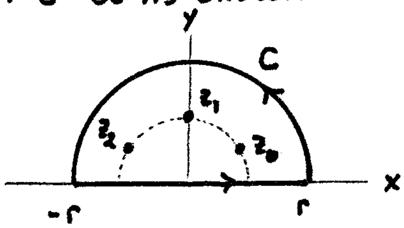
$z_0 = e^{\pi i/6}$, $z_1 = e^{\pi i/2}$, $z_2 = e^{5\pi i/6}$ ARE THE ONLY SINGULARITIES WITH $\mathrm{Im} > 0$.

$$\operatorname*{Res}_{z=z_i} f(z) = \frac{z_i^2}{6z_i^5} \quad \text{SO} \quad \operatorname*{Res}_{z=z_0} f(z) = \tfrac{1}{6}e^{-\pi i/2} = -\tfrac{1}{6}i$$

$$\operatorname*{Res}_{z=z_1} f(z) = \tfrac{1}{6}i$$

$$\operatorname*{Res}_{z=z_2} f(z) = -\tfrac{1}{6}i$$

CHOOSE $r > 1$ AND LET C BE AS SHOWN :



$$\oint_C \frac{z^2}{z^6+1}\,dz = 2\pi i\left[-\tfrac{1}{6}i + \tfrac{1}{6}i - \tfrac{1}{6}i\right] = \frac{\pi}{3}$$

THUS,

$$\int_{-r}^{r} \frac{x^2}{x^6+1}\,dx = \frac{\pi}{3} - \int_{C_r} \frac{z^2}{z^6+1}\,dz$$

SO

$$\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\,dx = \frac{\pi}{3} - \lim_{r\to\infty} \int_{C_r} \frac{z^2}{z^6+1}\,dz$$

NOW WE SHOW THAT THE LIMIT IS ZERO.

$$C_r : z(t) = re^{it}, \quad 0 \le t \le \pi$$

FOR z ON C_r,

$$\left| \frac{z^2}{z^6+1} \right| = \frac{|z|^2}{|z^6+1|} \leq \frac{|z|^2}{|z|^6-1} = \frac{r^2}{r^6-1} \qquad (=M)$$

SINCE THE LENGTH OF C_r IS $L = \pi r$,

$$\left| \int_{C_r} \frac{z^2}{z^6+1}\,dz \right| \leq ML = \left(\frac{r^2}{r^6-1}\right)(\pi r) = \frac{\pi r^3}{r^6-1} \to 0$$

AS $r \to \infty$

THUS,

$$\lim_{r\to\infty} \int_{C_r} \frac{z^2}{z^6+1}\,dz = 0$$

SO

$$\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\,dx = \frac{\pi}{3} .$$