INTEGRATION BY PARTS

RECALL THE PRODUCT RULE FOR DERIVATIVES : $u = u(x)$, $v = v(x)$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

INTEGRATE BOTH SIDES :

$$\int u\frac{dv}{dx}\,dx = \int \frac{d}{dx}(uv)\,dx - \int v\frac{du}{dx}\,dx$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

SHORTHAND NOTATION : THE INTEGRATION BY PARTS FORMULA :

$$\boxed{\int u\,dv = uv - \int v\,du}$$

EXAMPLES :

1. $\int x \sin x\,dx$: IDENTIFY SOMETHING TO CALL u . THE REST IS dv. COMPUTE du BY DIFFERENTIATION AND v BY INTEGRATION. PLUG IN THE INTEGRATION BY PARTS FORMULA AND HOPE THAT THE NEW INTEGRAL IS EASIER THAN THE ORIGINAL ONE.

   GENERALLY TRY TO CHOOSE u TO BE SOMETHING THAT SIMPLIFIES WHEN YOU DIFFERENTIATE IT.

$$\int x \sin x \, dx = (x)(-\cos x) - \int (-\cos x)\, dx$$

$u = x \qquad dv = \sin x\, dx$

$du = dx \qquad v = -\cos x$

$$= -x\cos x + \int \cos x \, dx$$

$$= -x\cos x + \sin x + C$$

2. $\int x e^x dx = xe^x - \int e^x dx = xe^x - e^x + C$

$u = x \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

3. $\int x^2 e^{-x} dx = (x^2)(-e^{-x}) - \int (-e^{-x})(2x\,dx)$

$u = x^2 \qquad dv = e^{-x} dx$

$du = 2x\,dx \qquad v = -e^{-x}$

$$= -x^2 e^{-x} + 2\int x e^{-x} dx$$

$u = x \qquad dv = e^{-x} dx$

$du = dx \qquad v = -e^{-x}$

$$= -x^2e^{-x} + 2\left[-xe^{-x} - \int -e^{-x} dx\right]$$

$$= -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x} dx$$

$$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

4. $\int \ln x \, dx$

$u = \ln x \quad dv = dx$
$du = \frac{1}{x} dx \quad v = x$

$$\int \ln x \, dx = x \ln x - \int x \left(\frac{1}{x} dx \right) = x \ln x - \int 1 \, dx$$
$$= x \ln x - x + C$$

5. $\int \arctan x \, dx$

$u = \arctan x \quad dv = dx$
$du = \frac{1}{1+x^2} dx \quad v = x$

$$\int \arctan x \, dx = x \arctan x - \int x \left(\frac{1}{1+x^2} dx \right)$$
$$= x \arctan x - \int \frac{1}{1+x^2} x \, dx$$
$$= x \arctan x - \frac{1}{2} \int \frac{1}{1+x^2} (2x \, dx)$$

$u = 1+x^2$ (DIFFERENT u !)
$du = 2x \, dx$

$$= x \arctan x - \frac{1}{2} \int \frac{1}{u} du$$
$$= x \arctan x - \frac{1}{2} \ln |u| + C$$
$$= x \arctan x - \frac{1}{2} \ln |1+x^2| + C$$
$$= x \arctan x - \frac{1}{2} \ln (1+x^2) + C$$

6. $$\int_{\sqrt{e}}^{e} \frac{\ln x}{x^2}\,dx = \int_{\sqrt{e}}^{e} (\ln x)x^{-2}dx$$

$$u = \ln x \qquad dv = x^{-3}dx$$
$$du = \frac{1}{x}dx \qquad v = -\frac{1}{2}x^{-2}$$

$$= -\frac{1}{2}x^{-2}\ln x \Big|_{\sqrt{e}}^{e} - \int_{\sqrt{e}}^{e} (-\tfrac{1}{2}x^{-2})(x^{-1}dx)$$

$$= -\frac{1}{2}\frac{\ln x}{x^2}\Big|_{\sqrt{e}}^{e} + \frac{1}{2}\int_{\sqrt{e}}^{e} x^{-3}dx$$

$$= -\frac{1}{2}\left[\frac{\ln e}{e^2} - \frac{\ln\sqrt{e}}{(\sqrt{e})^2}\right] + \frac{1}{2}\left(-\frac{1}{2}x^{-2}\Big|_{\sqrt{e}}^{e}\right)$$

$$= -\frac{1}{2}\left[\frac{1}{e^2} - \frac{\frac{1}{2}}{e}\right] - \frac{1}{4}\left[\frac{1}{e^2} - \frac{1}{e}\right]$$

$$= -\frac{1}{2}\left(\frac{1}{e^2}\right) + \frac{1}{4}\left(\frac{1}{e}\right) - \frac{1}{4}\left(\frac{1}{e^2}\right) + \frac{1}{4}\left(\frac{1}{e}\right)$$

$$= -\frac{3}{4}\left(\frac{1}{e^2}\right) + \frac{1}{2}\left(\frac{1}{e}\right)$$

7. $$\int \sin\sqrt{x}\,dx$$

THIS ONE IS TRICKY. FIRST MAKE THE (NONOBVIOUS) SUBSTITUTION $t = \sqrt{x}$ SO $dt = \frac{1}{2\sqrt{x}}dx$, I.E., $dx = 2\sqrt{x}\,dt = 2t\,dt$. THEN

$$\int \sin\sqrt{x}\,dx = \int \sin t\,(2t\,dt)$$
$$= 2\int t\sin t\,dt$$

NOW, INTEGRATION BY PARTS :

$$\int \sin\sqrt{x}\,dx = 2\int t\sin t\,dt$$

$$u = t \qquad dv = \sin t\,dt$$

$$du = dt \qquad v = -\cos t$$

$$= 2\left(-t\cos t + \int \cos t\,dt\right)$$

$$= -2t\cos t + 2\sin t + C$$

$$= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$$

8. $\int e^x \cos x\,dx$

NO OBVIOUS CHOICE FOR u (SOMETHING THAT SIMPLIFIES WHEN YOU DIFFERENTIATE IT) THIS TIME. JUST MAKE SOME CHOICE, E.G.,

$$u = e^x \qquad dv = \cos x\,dx$$

$$du = e^x dx \qquad v = \sin x$$

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

DOESN'T SEEM TO HAVE HELPED MUCH. JUST SWAPPED A COS FOR A SIN.

HOWEVER, WATCH WHAT HAPPENS WHEN THE INTEGRATION BY PARTS FORMULA IS APPLIED TO THE NEW INTEGRAL.

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

$u = e^x \qquad dv = \sin x\,dx$

$du = e^x dx \qquad v = -\cos x$

$$= e^x \sin x - \left(-e^x \cos x + \int e^x \cos x\,dx\right)$$

$$\int e^x \cos x\,dx = e^x \sin x + e^x \cos x - \int e^x \cos x\,dx$$

$$+ \int e^x \cos x\,dx \qquad\qquad + \int e^x \cos x\,dx$$

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x$$

$$\int e^x \cos x\,dx = \frac{1}{2}\left(e^x \sin x + e^x \cos x\right) + C$$

9. PROVE THE <u>REDUCTION</u> <u>FORMULA</u> : FOR $n \geq 2$,

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

$$\int \cos^n x\,dx = \int \cos^{n-1} x \cos x\,dx$$

$u = \cos^{n-1} x \qquad dv = \cos x\,dx$

$du = (n-1)\cos^{n-2} x\,(-\sin x)\,dx \qquad v = \sin x$

$$\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx$$

$$\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$

$$+ (n-1)\int \cos^n x\,dx \qquad\qquad + (n-1)\int \cos^n x\,dx$$

$$\underbrace{(1+(n-1))}_{n}\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

AS REQUIRED.

10. USE THE REDUCTION FORMULA TO FIND $\int \cos^4 x\,dx$

$$\int \cos^4 x\,dx = \frac{1}{4}\cos^{4-1} x \sin x + \frac{4-1}{4}\int \cos^{4-2} x\,dx$$

$$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\underbrace{\int \cos^2 x\,dx}_{\text{REDUCTION FORMULA AGAIN !}}$$

$$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\left[\frac{1}{2}\cos^{2-1} x \sin x + \frac{2-1}{2}\int \cos^{2-2} x\,dx\right]$$

$$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}\int 1\,dx$$

$$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + C$$