# INTEGRATION BY SUBSTITUTION

INTEGRATION (FINDING ANTIDERIVATIVES) CAN BE A VERY TRICKY BUSINESS, E.G.,

$$\int e^x \, dx$$

IS COMPLETELY TRIVIAL, BUT THERE IS A PRECISE SENSE IN WHICH

$$\int e^{x^2} dx$$

IS IMPOSSIBLE.

> $e^{x^2}$ HAS NO "ELEMENTARY ANTIDERIVATIVE". THINK OF ALL THE FUNCTIONS YOU'VE EVER SEEN (POLYNOMIALS, LOGARITHMIC FUNCTIONS, EXPONENTIAL FUNCTIONS, TRIGONOMETRIC OR INVERSE TRIGONOMETRIC FUNCTIONS, ...) AND COMBINE THEM (SUMS, PRODUCTS, QUOTIENTS, COMPOSITIONS, ...) ANYWAY YOU WANT FOR AS LONG AS YOU WANT. YOU WILL <u>NEVER</u> WRITE DOWN AN ANTIDERIVATIVE FOR $e^{x^2}$.

EVEN WITH A HUGE TABLE OF INTEGRALS IN FRONT OF YOU IT CAN BE TRICKY.

> THERE IS A RELATIVELY SMALL TABLE WITH 122 ENTRIES INSIDE THE FRONT AND BACK COVERS OF YOUR TEXTBOOK. THE INTEGRAL

$$\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$$

DOES NOT APPEAR TO BE SIMILAR TO ANY OF THESE 122 ENTRIES.

HOWEVER, WITH A FEW "TRICKS OF THE TRADE" THIS INTEGRAL CAN BE OBTAINED FROM THE TABLE.

THE MOST BASIC OF THESE TRICKS OF THE TRADE IS

<u>INTEGRATION</u> <u>BY</u> <u>SUBSTITUTION</u>

<u>MOTIVATION</u> : HERE'S AN EASY DERIVATIVE :

$$(\sin(x^2+1))' = \cos(x^2+1)(x^2+1)'$$
$$= \cos(x^2+1)2x$$

THOUGHT OF AS AN INTEGRATION FORMULA THIS SAYS

$$\int \cos(x^2+1)2x\,dx = \sin(x^2+1) + C$$

WITHOUT HAVING DONE THE DERIVATIVE FIRST THIS INTEGRAL MIGHT NOT BE SO EASY TO WRITE OUT.

" SUBSTITUTION " IS A LITTLE NOTATIONAL TRICK FOR MAKING IT EASY.

$$\int \cos \underbrace{(x^2+1)}_{u=x^2+1} \underbrace{2x\,dx}_{\frac{du}{dx}=2x}$$

"DIFFERENTIAL FORM" IS

$du = 2x\,dx$

$$= \int \cos u \, du$$

$$= \sin u + C$$

$$= \sin(x^2+1) + C$$

<u>SUBSTITUTION</u> <u>TECHNIQUE</u> : FIND SOMETHING IN THE INTEGRAND TO CALL $u$ THAT SIMPLIFIES THE APPEARENCE OF THE INTEGRAL AND WHOSE $du = \frac{du}{dx}\,dx$ IS ALSO PRESENT AS A FACTOR.

<u>NOTE</u> : THE TECHNIQUE WOULD HAVE FAILED FOR $\int \cos(x^2+1)\,dx$, WHICH, IN FACT, IS JUST AS IMPOSSIBLE AS $\int e^{x^2}dx$.

<u>EXAMPLES</u> :

1. $\int \sec^2(\frac{1}{3}x^3)\,x^2\,dx = \int \sec^2 u\,du = \tan u + C$

$= \tan(\frac{1}{3}x^3) + C$

$u = \frac{1}{3}x^3$

$\frac{du}{dx} = x^2$

$du = x^2\,dx$

2. $\int \sqrt{\sin x}\, \cos x\, dx \quad = \int \sqrt{u}\, du = \int u^{\frac{1}{2}} du$

$u = \sin x$ $\quad = \frac{2}{3} u^{\frac{3}{2}} + C$

$\frac{du}{dx} = \cos x$ so $du = \cos x\, dx$ $\quad = \frac{2}{3} (\sin x)^{\frac{3}{2}} + C$

3. $\int 4x^3 (x^4+3)^{10} dx \quad = \int (x^4+3)^{10} 4x^3 dx = \int u^{10} du$

$u = x^4 + 3$ $\quad = \frac{1}{11} u^{11} + C$

$\frac{du}{dx} = 4x^3$ so $du = 4x^3 dx$ $\quad = \frac{1}{11} (x^4+3)^{11} + C$

4. $\int x^3 (x^4+3)^{10} dx \quad = \frac{1}{4} \int (x^4+3)^{10} 4x^3 dx = \frac{1}{4} \int u^{10} du$

$u = x^4 + 3$ $\quad = \frac{1}{4} \frac{1}{11} u^{11} + C$

$du = 4x^3 dx$ $\quad = \frac{1}{44} (x^4+3)^{11} + C$

NOTE : IF A CONSTANT FACTOR IS MISSING, INSERT IT AND CANCEL IT OUTSIDE THE INTEGRAL (CONSTANTS CAN BE MOVED IN OR OUT OF INTEGRALS).

5. $\int \frac{\sin \sqrt{t}}{\sqrt{t}} dt \quad = 2 \int \sin \sqrt{t} \left(\frac{1}{2\sqrt{t}}\right) dt = 2 \int \sin u\, du$

$\quad = -2 \cos u + C$

$u = \sqrt{t}$ $\quad = -2 \cos \sqrt{t} + C$

$du = \frac{1}{2\sqrt{t}} dt$

6. $\int \cos(7-3x)\,dx = -\frac{1}{3}\int \cos(7-3x)(-3dx) = -\frac{1}{3}\int \cos u\,du$

$u = 7-3x$
$du = -3\,dx$

$= -\frac{1}{3}\sin u + C$

$= -\frac{1}{3}\sin(7-3x) + C$

7. $\int \frac{1}{1+16x^2}\,dx = \int \frac{1}{1+(4x)^2}\,dx = \frac{1}{4}\int \frac{1}{1+(4x)^2}(4dx) =$

$u = 4x$
$du = 4dx$

$\frac{1}{4}\int \frac{1}{1+u^2}\,du =$

$\frac{1}{4}\,\text{ARCTAN}\,u + C =$

$\frac{1}{4}\,\text{ARCTAN}\,(4x) + C$

8. $\int \frac{t}{\sqrt{1-t^4}}\,dt = \int \frac{t}{\sqrt{1-(t^2)^2}}\,dt = \frac{1}{2}\int \frac{1}{\sqrt{1-(t^2)^2}}(2t\,dt)$

$u = t^2$
$du = 2t\,dt$

$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du$

$= \frac{1}{2}\,\text{ARCSIN}\,u + C$

$= \frac{1}{2}\,\text{ARCSIN}\,(t^2) + C$

9. $\int \frac{x}{\sqrt{x+1}}\,dx$

THIS ONE IS A BIT TRICKIER TO SPOT. HERE'S THE IDEA :

$u = x+1 \implies x = u-1$
$du = dx$

$\int \frac{x}{\sqrt{x+1}}\,dx = \int \frac{u-1}{\sqrt{u}}\,du$ AND NOW WE CAN DIVIDE OUT

$= \int \frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\,du$

$$= \int u^{\frac{1}{2}} - u^{-\frac{1}{2}} du$$

$$= \frac{2}{3} u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C$$

$$= \frac{2}{3}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C$$

10. $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{\cos x}(-\sin x \, dx) =$

$u = \cos x$
$du = -\sin x \, dx$

$-\int \frac{1}{u} du =$

$-\ln|u| + C =$

$-\ln|\cos x| + C =$

$\ln|\sec x| + C$

THIS ONE IS WORTH REMEMBERING :

$$\boxed{\int \tan x \, dx = \ln|\sec x| + C}$$

SIMILARLY,

$$\boxed{\int \cot x \, dx = \ln|\sin x| + C}$$