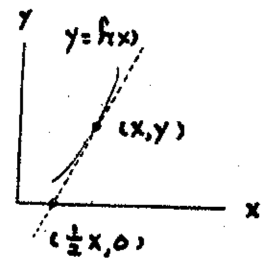
INTRODUCTION TO DIFFERENTIAL EQUATIONS :

BEGIN WITH SOME MOTIVATION.

1. CONSIDER THE FOLLOWING GEOMETRICAL PROBLEM :

FIND A FUNCTION $y = f(x)$ ON $x > 0$
WHOSE TANGENT LINE AT ANY POINT (x, y)
INTERSECTS THE X-AXIS AT $(\frac{1}{2}x, 0)$



COMPUTE THE SLOPE OF THE TANGENT LINE AT (x, y) IN TWO WAYS :

$$\frac{dy}{dx} = \frac{y - 0}{x - \frac{1}{2}x}$$

$$\boxed{\frac{dy}{dx} = \frac{2y}{x}}$$

NOTE THAT THIS DOES NOT TELL US WHAT $y = f(x)$ IS, BUT ONLY TELLS US WHAT CONDITION IT MUST SATISFY IN ORDER TO BE A SOLUTION TO OUR PROBLEM.

2. RADIOACTIVE SUBSTANCES (AT LEAST UNDER A CERTAIN RANGE OF CONDITIONS) DECAY AT A RATE PROPORTIONAL TO THE AMOUNT OF THE SUBSTANCE PRESENT.

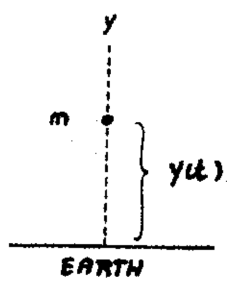
$Q(t)$ = AMOUNT PRESENT AT TIME t

THEN FOR SOME CONSTANT $k < 0$,

$$\boxed{\frac{dQ}{dt} = kQ}$$

SAME STORY AS IN #1. THIS DOESN'T TELL US WHAT $Q(t)$ IS, BUT ONLY GIVES US A CONDITION $Q(t)$ MUST SATISFY WHICH AGAIN INVOLVES THE DERIVATIVE OF THE UNKNOWN FUNCTION $Q(t)$.

3. LAST TERM WE SAW THAT AN OBJECT FALLING FREELY IN THE EARTH'S GRAVITATIONAL FIELD EXPERIENCES A CONSTANT DOWNWARD ACCELERATION.



$$m\frac{d^2y}{dt^2} = -mg \qquad \text{(NEWTON'S 2}^{\text{ND}}\text{ LAW)}$$

OR

$$\frac{dv}{dt} = -g \qquad \left(v = \frac{dy}{dt}\right)$$

IF IT ISN'T FALLING "FREELY" BUT IS SUBJECT TO AIR RESISTANCE (GENERALLY ASSUMED PROPORTIONAL TO THE VELOCITY), THEN

$$m \frac{dv}{dt} = -mg - cv$$

FOR SOME CONSTANT $c > 0$.

$$\boxed{\frac{dv}{dt} + \frac{c}{m} v = -g}$$

ONCE AGAIN WE HAVE AN EQUATION FOR THE UNKNOWN FUNCTION $v(t)$ INVOLVING THAT FUNCTION AND ITS DERIVATIVE.

IF WE WROTE THIS IN TERMS OF THE HEIGHT $y(t)$ IT WOULD READ

$$\boxed{\frac{d^2y}{dt^2} + \frac{c}{m} \frac{dy}{dt} = -g}$$

AN <u>ORDINARY</u> <u>DIFFERENTIAL</u> <u>EQUATION</u> (<u>ODE</u>) IS AN EQUATION TO BE SOLVED FOR SOME UNKNOWN FUNCTION THAT INVOLVES THAT FUNCTION AND ITS DERIVATIVES.

THE <u>ORDER</u> OF THE ODE IS THE ORDER OF THE HIGHEST ORDER DERIVATIVE THAT APPEARS IN THE EQUATION.

E.G., $\frac{dy}{dx} = \frac{2y}{x}$ AND $\frac{dQ}{dt} = kQ$ ARE

1^{ST} ORDER, BUT $\frac{d^2y}{dt^2} + c\frac{dy}{dt} = -g$ IS 2^{ND} ORDER

A <u>SOLUTION</u> TO AN ODE IS JUST A FUNCTION THAT SATISFIES THE EQUATION, E.G.,

1. $y' = y$

SOLUTIONS : $y = e^x$

$y = 3e^x$

$\vdots$

$y = Ae^x$ FOR ANY CONSTANT A

2. $y'' + y = 0$

SOLUTIONS : $y = \sin x$

$y = \cos x$

$y = \sin x + \cos x$

$y = A \sin x$ FOR ANY CONSTANT A

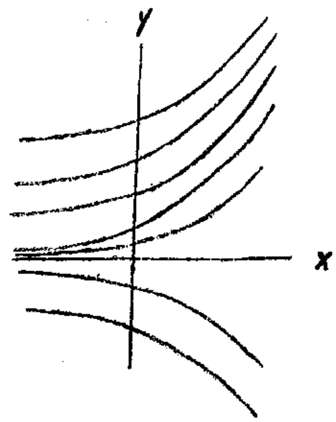
$y = B \cos x$ FOR ANY CONSTANT B

$y = A \sin x + B \cos x$

GENERALLY, A DIFFERENTIAL EQUATION HAS INFINITELY MANY SOLUTIONS, BUT IT MIGHT ONLY HAVE ONE $\left((y')^2 + y^2 = 0 \right)$ OR IT MIGHT HAVE NONE $\left((y')^2 + y^2 = -1 \right)$.

THE GRAPHS OF THE SOLUTIONS TO A DIFFERENTIAL EQUATION ARE CALLED ITS <u>INTEGRAL CURVES</u>.

E.G., THE INTEGRAL CURVES OF $y' = y$ ARE



$y = Ae^x$ FOR VARIOUS CONSTANTS A

<u>NOTE</u>: FOR THIS 1^{ST} ORDER EQUATION THERE IS EXACTLY ONE INTEGRAL CURVE THROUGH EVERY POINT (x_0, y_0) IN THE PLANE.

<u>GENERAL 1^{ST} ORDER ODE</u>: $\qquad y' = f(x,y)$

E.G., $y' = y$, $y' = x^2 + y^2$

<u>PROBLEM</u>: FIND ALL THE SOLUTIONS.

<u>GENERAL 1^{ST} ORDER INITIAL VALUE PROBLEM</u>: $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$

<u>PROBLEM</u>: FIND THE SOLUTION TO $y' = f(x,y)$ WHOSE GRAPH GOES THROUGH THE POINT (x_0, y_0).

MOST OF THESE PROBLEMS ARE EXTREMELY DIFFICULT. WE WILL CONSIDER JUST THREE SPECIAL CASES THAT ARE MANAGEABLE AND ACTUALLY ARISE OFTEN IN APPLICATIONS.

1. ("SIMPLE" INTEGRATION)

$$\frac{dy}{dx} = f(x)$$

NO y ON THE RIGHT HAND SIDE. JUST INTEGRATE BOTH SIDES WITH RESPECT TO x.

EXAMPLES :

1. SOLVE : $\frac{dy}{dx} = x\sqrt{x^2+4}$

INTEGRATE : $y = \int x\sqrt{x^2+4}\,dx$

$$= \frac{1}{2}\int (x^2+4)^{\frac{1}{2}}(2x\,dx)$$

$$= \frac{1}{2}\,\frac{2}{3}(x^2+4)^{\frac{3}{2}} + C$$

$$y = \frac{1}{3}(x^2+4)^{\frac{3}{2}} + C$$

2. SOLVE THE INITIAL VALUE PROBLEM : $\begin{cases} \frac{dy}{dx} = x\sqrt{x^2+4} \\ y(-4) = 0 \end{cases}$

IN THE FIRST PROBLEM WE FOUND ALL OF THE SOLUTIONS TO

$\frac{dy}{dx} = x\sqrt{x^2+4}$:

$$y(x) = \frac{1}{3}(x^2+4)^{\frac{3}{2}} + C$$

NOW WE SOLVE FOR THE C THAT WILL ENSURE

$$0 = y(-4) = \frac{1}{3}((-4)^2+4)^{\frac{3}{2}} + C$$

$$0 = \frac{1}{3}(20)^{\frac{3}{2}} + C$$

$$0 = \frac{1}{3}\cdot 20\sqrt{20} + C$$

$$0 = \frac{40\sqrt{5}}{3} + C$$

$$C = -\frac{40\sqrt{5}}{3}$$

$$y(x) = \frac{1}{3}(x^2+4)^{\frac{3}{2}} - \frac{40\sqrt{5}}{3}$$

2. (LINEAR EQUATIONS)

A 1^{ST} ORDER LINEAR ODE IS ONE THAT CAN BE WRITTEN IN THE FORM

$$y' + P(x)y = Q(x)$$

FOR SOME (KNOWN) FUNCTIONS $P(x)$ AND $Q(x)$, E.G.,

$$y' + 3y = e^{-2x}$$

$$\frac{dv}{dt} + \frac{c}{m}v = -g$$

$$\frac{dQ}{dt} = kQ \qquad \left(\frac{dQ}{dt} - kQ = 0 \right)$$

$$\frac{dy}{dx} = \frac{2y}{x} \qquad \left(\frac{dy}{dx} - \frac{2}{x}y = 0 \right)$$

NOTE THAT YOU CAN'T SOLVE THESE BY JUST "INTEGRATING BOTH SIDES" SINCE THIS WOULD BURY THE UNKNOWN FUNCTION INSIDE AN INTEGRAL SIGN.

THERE IS, HOWEVER, A "TRICK" WHICH GETS AROUND THIS PROBLEM.

CONSIDER

$$y' + 3y = e^{-2x}$$

LOOK WHAT HAPPENS WHEN YOU MULTIPLY BOTH SIDES BY

$$\mu(x) = e^{3x}$$

(I'LL EXPLAIN WHERE THIS CAME FROM SHORTLY)

$$\underbrace{e^{3x}y' + 3e^{3x}y}_{\text{PRODUCT RULE FOR}} = e^{x}$$

$$(e^{3x}y)' = e^{x}$$

NOW WE CAN INTEGRATE BOTH SIDES WITHOUT BURYING y INSIDE AN INTEGRAL SIGN :

$$\int (e^{3x}y)'\,dx = \int e^{x}dx$$

$$e^{3x}y = e^{x} + c$$

$$y = e^{-2x} + ce^{-3x}$$

SO THE QUESTION IS, "WHERE DID THE MAGICAL $\mu(x) = e^{3x}$ COME FROM ? "

<u>NOTE</u> :

$$y' + 3y = e^{-2x}$$

$$y' + P(x)y = Q(x)$$

HERE $P(x) = 3$ SO $\mu(x) = e^{\int P(x)dx}$

(JUST ONE ANTIDERIVATIVE NEEDED IN THE EXPONENT)

WE WILL NOW SHOW THAT FOR ANY LINEAR EQUATION

$$y' + P(x)y = Q(x)$$

THE FUNCTION

$$\mu(x) = e^{\int P(x)dx}$$

(CALLED AN <u>INTEGRATING</u> <u>FACTOR</u> FOR THE EQUATION) WORKS JUST

THE WAY e^{3x} DID IN THE EXAMPLE, I.E., MULTIPLYING BOTH SIDES OF $y' + P(x)y = Q(x)$ BY $\mu(x)$ TURNS THE EQUATION INTO

$$(\mu(x)y)' = \mu(x)Q(x)$$

WHICH CAN JUST BE INTEGRATED AND SOLVED FOR y.

HERE'S THE REASON :

$$y' + P(x)y = Q(x)$$

$$\underbrace{\mu(x)y' + \mu(x)P(x)y} = \mu(x)Q(x)$$

$$\begin{aligned}(\mu(x)y)' &= \mu(x)y' + \mu'(x)y \\ &= \mu(x)y' + \left(e^{\int P(x)dx}\right)'y \\ &= \mu(x)y' + e^{\int P(x)dx}\left(\int P(x)dx\right)'y \\ &= \mu(x)y' + \mu(x)P(x)y\end{aligned}$$

$$(\mu(x)y)' = \mu(x)Q(x)$$

<u>EXAMPLES</u> :

1. SOLVE : $y' - 2xy = x$

HERE $P(x) = -2x$ SO

$$\mu(x) = e^{\int P(x)dx} = e^{\int -2x\,dx} = e^{-x^2}$$

SO

$$e^{-x^2}[\,y' - 2xy = x\,]$$

$$e^{-x^2}y' - 2xe^{-x^2}y = xe^{-x^2}$$

$$(e^{-x^2}y)' = xe^{-x^2}$$

$$e^{-x^2}y = \int xe^{-x^2}dx$$

$$e^{-x^2}y = -\tfrac{1}{2}\int e^{-x^2}(-2x\,dx)$$

$$e^{-x^2}y = -\tfrac{1}{2}e^{-x^2} + c$$

$$y = -\tfrac{1}{2} + ce^{x^2}$$

2. SOLVE $\begin{cases} x^2y' + xy = x\sin x \\ y(\pi) = 2 \end{cases}$ ON $x > 0$

ON $x > 0$ WE CAN DIVIDE THE EQUATION BY x^2 TO GET

$$y' + \frac{1}{x}y = \frac{\sin x}{x}$$

SO

$$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = e^{\ln x} = x$$

$$x\left[\,y' + \frac{1}{x}y = \frac{\sin x}{x}\,\right]$$

$$xy' + y = \sin x$$

$$(xy)' = \sin x$$

$$xy = \int \sin x \, dx$$

$$xy = -\cos x + c$$

$$y = -\frac{\cos x}{x} + \frac{c}{x}$$

TO ENSURE $y(\pi) = 2$,

$$2 = y(\pi) = -\frac{\cos \pi}{\pi} + \frac{c}{\pi}$$

$$2 = \frac{1}{\pi} + \frac{c}{\pi}$$

$$2\pi = 1 + c$$

$$c = 2\pi - 1$$

SO

$$y = -\frac{\cos x}{x} + \frac{2\pi - 1}{x}$$

3. SOLVE $\begin{cases} \frac{dv}{dt} + \frac{c}{m} v = -g \\ v(0) = v_0 \end{cases}$ AND INVESTIGATE $\lim_{t \to \infty} v(t)$.

$$\mu(t) = e^{\int \frac{c}{m} dt} = e^{\frac{c}{m} t}$$

$$e^{\frac{c}{m} t} \left[\frac{dv}{dt} + \frac{c}{m} v = -g \right]$$

$$e^{\frac{c}{m} t} \frac{dv}{dt} + \frac{c}{m} e^{\frac{c}{m} t} v = -g e^{\frac{c}{m} t}$$

$$\left(e^{\frac{c}{m} t} v \right)' = -g e^{\frac{c}{m} t}$$

$$e^{\frac{c}{m}t} v = \int -g e^{\frac{c}{m}t}\, dt$$

$$e^{\frac{c}{m}t} v = -\frac{gm}{c} e^{\frac{c}{m}t} + D$$

("c" IS ALREADY IN USE)

$$v(t) = -\frac{gm}{c} + D e^{-\frac{c}{m}t}$$

$$v_0 = v(0) = -\frac{gm}{c} + D$$

$$v_0 + \frac{gm}{c} = D$$

THUS,

$$v(t) = -\frac{gm}{c} + \left(v_0 + \frac{gm}{c}\right) e^{-\frac{c}{m}t}$$

SINCE c AND m ARE <u>POSITIVE</u> CONSTANTS,

$$\lim_{t \to \infty} v(t) = -\frac{gm}{c}$$

$$= \underline{\text{TERMINAL VELOCITY}}$$

3. (<u>SEPARABLE</u> <u>EQUATIONS</u>)

A 1ST <u>ORDER</u> <u>SEPARABLE</u> <u>ODE</u> IS ONE THAT CAN BE WRITTEN IN THE FORM

$$h(y)\frac{dy}{dx} = g(x)$$

OR, IN "DIFFERENTIAL FORM"

$$h(y)\,dy = g(x)\,dx$$

(VARABLES "SEPARATED" ON OPPOSITE SIDES OF THE EQUATION)

E.G.,

$$\frac{dy}{dx} = \frac{2x^2y+y}{x} \qquad \left(\frac{dy}{dx} = \frac{(2x^2+1)y}{x} \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{2x^2+1}{x} \Rightarrow \frac{1}{y}\,dy = \frac{2x^2+1}{x}\,dx \right)$$

$$\frac{dy}{dx} = \frac{2y}{x} \qquad \left(\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} \Rightarrow \frac{1}{y}\,dy = \frac{2}{x}\,dx \right)$$

$$\frac{dQ}{dt} = kQ \qquad \left(\frac{1}{Q}\frac{dQ}{dt} = k \Rightarrow \frac{1}{Q}\,dQ = k\,dt \right)$$

TO SEE HOW SUCH EQUATIONS CAN BE SOLVED:

$$h(y)\frac{dy}{dx} = g(x)$$

LET $H(y)$ AND $G(x)$ BE ANTIDERIVATIVES FOR $h(y)$ AND $g(x)$.

$$\frac{dH}{dy} = h(y) \qquad \frac{dG}{dx} = g(x)$$

EQUATION BECOMES

$$\frac{dH}{dy}\frac{dy}{dx} = \frac{dG}{dx}$$

NOW RECALL THE CHAIN RULE :

$$H = H(y) \quad \text{AND} \quad y = y(x) \Rightarrow$$

$$H = H(y(x))$$

$$\frac{dH}{dx} = \frac{dH}{dy}\frac{dy}{dx}$$

THUS, THE EQUATION SAYS

$$\frac{d}{dx} H(y) = \frac{d}{dx} G(x)$$

INTEGRATE BOTH SIDES WITH RESPECT TO X :

$$H(y) = G(x) + C$$

NOTICE THAT THIS IS EXACTLY WHAT YOU WOULD GET BY FORMALLY INTEGRATING BOTH SIDES OF

$$h(y)dy = g(x)dx$$

$$\int h(y)dy = \int g(x)dx$$

$$H(y) = G(x) + C$$

AS A PRACTICAL MATTER THIS IS HOW THE EQUATIONS ARE ACTUALLY SOLVED.

EXAMPLES :

1. $$\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$$
$$= (1+x^2) + y^2(1+x^2)$$
$$= (1+x^2)(1+y^2)$$

$$\frac{1}{1+y^2}\frac{dy}{dx} = 1 + x^2$$

$$\frac{1}{1+y^2}\,dy = (1+x^2)\,dx$$

$$\int \frac{1}{1+y^2}\,dy = \int (1+x^2)\,dx$$

$$\arctan y = x + \tfrac{1}{3}x^3 + C \qquad \text{(IMPLICIT SOLUTION)}$$

$$y = \tan\left(x + \tfrac{1}{3}x^3 + C\right) \qquad \text{(EXPLICIT SOLUTION)}$$

2. $$\frac{dy}{dx} = \frac{2x^2y + y}{x} = \frac{y(2x^2+1)}{x} \qquad (x \neq 0)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x^2+1}{x}$$

NOTE : THIS ASSUMES $y \neq 0$. HOWEVER, $y = 0$ IS A SOLUTION TO THE EQUATION.

$$\frac{1}{y}\,dy = \left(2x + \frac{1}{x}\right)dx$$

$$\int \frac{1}{y}\,dy = \int \left(2x + \frac{1}{x}\right)dx$$

$$\ln|y| = x^2 + \ln|x| + C$$

$$\ln|y| - \ln|x| = x^2 + c$$

$$\ln\left|\frac{y}{x}\right| = x^2 + c$$

$$\left|\frac{y}{x}\right| = e^{x^2+c} = e^c e^{x^2}$$

$$\frac{y}{x} = (\pm e^c)e^{x^2}$$

$$\frac{y}{x} = k e^{x^2}$$

$$y = kxe^{x^2}$$

NOTE : $k = \pm e^c$ IS AN ARBITRARY NONZERO CONSTANT. HOWEVER, TAKING $k = 0$ GIVES THE SOLUTION $y = 0$ THAT WE ELIMINATED EARLIER.