INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS

THE DEFINITION IS SIMPLE ENOUGH : A PARTIAL DIFFERENTIAL EQUATION (PDE) IS AN EQUATION TO BE SOLVED FOR AN UNKNOWN FUNCTION OF TWO OR MORE VARIABLES THAT INVOLVES THIS FUNCTION AND ITS PARTIAL DERIVATIVES.

E.G., FIND ALL $u(x,y)$ SATISFYING

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

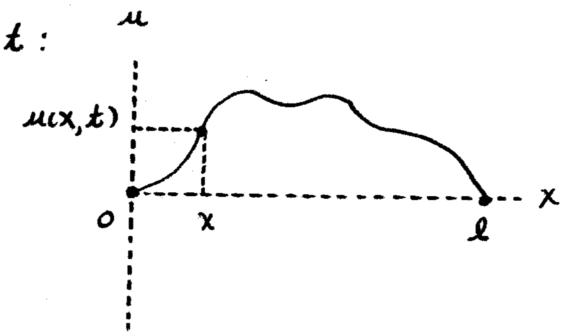
(SEE HOW MANY SOLUTIONS YOU CAN FIND " BY INSPECTION ")

THE STUDY OF SUCH EQUATIONS IS ENORMOUSLY COMPLICATED, HOWEVER, AND WE WILL SAY JUST A FEW THINGS ABOUT SOME VERY IMPORTANT EXAMPLES.

1. (1-DIMENSIONAL WAVE EQUATION)

CONSIDER A THIN, ELASTIC STRING THAT IS TIGHTLY STRETCHED AND THEN HELD FIXED AT ITS ENDPOINTS (E.G., A GUITAR STRING). SET IT IN MOTION SO THAT IT EXPERIENCES SMALL, VERTICAL VIBRATIONS,

" SNAPSHOT " AT TIME t :



$u(x,t)$ = HEIGHT OF THE STRING AT LOCATION x AND TIME t

PHYSICS IMPLIES

$u(x,t)$ SATISFIES THE 1-DIMENSIONAL WAVE EQUATION

$$\boxed{c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}}$$

WHERE c IS A CONSTANT THAT DEPENDS ON WHAT THE STRING IS MADE OF AND HOW TIGHTLY IT IS STRETCHED (DENSITY AND TENSION).

INTUITIVELY, THE VERTICAL ACCELERATION ($\frac{\partial^2 u}{\partial t^2}$) IS PROPORTIONAL TO THE CONCAVITY ($\frac{\partial^2 u}{\partial x^2}$) (AMOUNT OF " BENDING ").

<u>EXERCISE 1</u> : SHOW THAT

$$u(x,t) = \cos \frac{2\pi c t}{\ell} \sin \frac{2\pi x}{\ell}$$

IS A SOLUTION. THEN DETERMINE HOW THE STRING MUST BE SET IN MOTION TO MAKE IT MOVE THIS WAY, I.E., FIND THE " INITIAL DISPLACEMENT " $u(x,0)$ AND " INITIAL VELOCITY " $\frac{\partial u}{\partial t}(x,0)$. FINALLY, DESCRIBE (CAREFULLY) HOW THE STRING IS ACTUALLY MOVING (THE MOTION IS CALLED A <u>STANDING</u> <u>WAVE</u>). <u>HINT</u>: THINK ABOUT THE " SNAPSHOTS ".

NOTE : THERE IS ALSO A 2- DIMENSIONAL WAVE EQUATION

$$\boxed{c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial^2 u}{\partial t^2}}$$

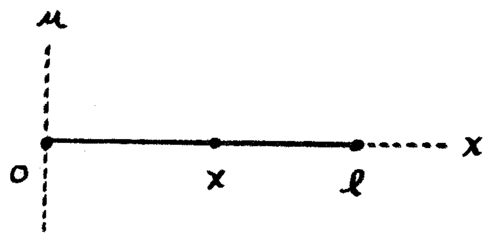
THAT DESCRIBES THE VIBRATIONS OF A MEMBRANE ("DRUM") AND A 3- DIMENSIONAL WAVE EQUATION

$$\boxed{c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial^2 u}{\partial t^2}}$$

SATISFIED, FOR EXAMPLE, BY THE COMPONENTS OF AN ELECTROMAGNETIC FIELD,

2. (1- DIMENSIONAL HEAT EQUATION)

CONSIDER A THIN, HOMOGENEOUS WIRE LYING ALONG THE X-AXIS BETWEEN $x = 0$ AND $x = \ell$.



PROVIDE THE WIRE WITH SOME INITIAL TEMPERATURE DISTRIBUTION (E.G., HOLD A CANDLE UNDER IT) AND THEN REMOVE THE HEAT SOURCE. THERMAL ENERGY FLOWS IN AN ATTEMPT TO ACHIEVE THERMAL EQUILIBRIUM.

$u(x, t)$ = TEMPERATURE AT LOCATION X AND TIME t

PHYSICS IMPLIES

$u(x,t)$ SATISFIES THE 1-DIMENSIONAL HEAT EQUATION

$$\boxed{c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}}$$

WHERE c IS A CONSTANT DETERMINED BY THE MATERIAL OF WHICH THE WIRE IS COMPOSED.

<u>EXERCISE 2</u> : SHOW THAT

$$u(x,t) = e^{-c^2\pi^2 t/\ell^2} \cos \frac{\pi x}{\ell}$$

IS A SOLUTION. DESCRIBE THE INITIAL TEMPERATURE DISTRIBUTION $u(x,0)$ AS WELL AS THE LONG TERM ($t \to \infty$) BEHAVIOR OF THE TEMPERATURE FUNCTION.

TEMPERATURE EVOLUTION IN A 2-DIMENSIONAL PLATE ARE DESCRIBED BY

$$\boxed{c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial u}{\partial t}}$$

AND IN A 3-DIMENSIONAL SOLID BY

$$\boxed{c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial u}{\partial t}}$$

SOME TERMINOLOGY AND NOTATION : THE LAPLACIAN ∇^2 :

3-DIMENSIONAL LAPLACIAN : $\nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$

2-DIMENSIONAL LAPLACIAN : $\nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$

1-DIMENSIONAL LAPLACIAN : $\nabla^2 u = \dfrac{\partial^2 u}{\partial x^2}$

WAVE EQUATION :

$$c^2 \nabla^2 u = \frac{\partial^2 u}{\partial t^2}$$

HEAT EQUATION :

$$c^2 \nabla^2 u = \frac{\partial u}{\partial t}$$

3. (LAPLACE EQUATION)

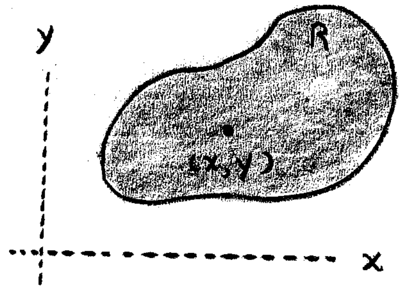
WHEN u DOES NOT DEPEND ON t BOTH THE WAVE EQUATION AND THE HEAT EQUATION REDUCE TO THE LAPLACE EQUATION

$$\boxed{\nabla^2 u = 0}$$

HERE ARE A FEW OF THE PHYSICAL SITUATIONS DESCRIBED BY THE LAPLACE EQUATION :

A. STEADY STATE (I.E., TIME INDEPENDENT) TEMPERATURE DISTRIBUTIONS

E.G., CONSIDER A THIN HOMOGENEOUS SLAB OF MATERIAL OCCUPYING A REGION R OF THE XY-PLANE.



IF THE SLAB IS IN THERMAL EQUILIBRIUM (TEMPERATURES DO NOT CHANGE) AND

$u(x,y)$ = TEMPERATURE AT LOCATION (x,y)

THEN $u(x,y)$ SATISFIES

$$\nabla^2 u = 0.$$

B. STATIC TRANSVERSE DISPLACEMENTS OF A MEMBRANE ("DRUM").

SUPPOSE THE REGION R SHOWN ABOVE IS A TIGHTLY STRETCHED SHEET OF RUBBER AND THAT ITS BOUNDARY IS A WIRE TO WHICH THE SHEET IS ATTACHED.

BEND THE WIRE SO THAT EACH POINT IS DISPLACED VERTICALLY SOME DISTANCE ABOVE OR BELOW THE XY-PLANE.

THE MEMBRANE IS NOW A CURVED SURFACE IN SPACE AND IF

$$u(x,y) = \text{HEIGHT OF THE MEMBRANE AT } (x,y)$$

THEN u SATISFIES

$$\nabla^2 u = 0.$$

C. GRAVITATIONAL OR ELECTROSTATIC POTENTIAL FUNCTIONS

<u>NOTE</u>: $\nabla^2 u = 0$ IS ALSO CALLED THE <u>POTENTIAL</u> <u>EQUATION</u> AND ITS SOLUTIONS ARE CALLED <u>HARMONIC</u> <u>FUNCTIONS</u>.

THE PROBLEM OF ACTUALLY <u>SOLVING</u> THESE EQUATIONS IS NO SIMPLE MATTER AND WE WILL DESCRIBE ONLY TWO OF THE MOST BASIC IDEAS AND ONLY FOR THE 1-DIMENSIONAL WAVE EQUATION.

1. THE D'ALEMBERT SOLUTION

2. SEPARATION OF VARIABLES

D'ALEMBERT'S SOLUTION TO THE WAVE EQUATION

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

INTRODUCE NEW VARIABLES :

$$v = x + ct \qquad w = x - ct$$

THEN

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial u}{\partial v} + \frac{\partial u}{\partial w} = u_v + u_w$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(u_v + u_w) = \frac{\partial u_v}{\partial x} + \frac{\partial u_w}{\partial x}$$

$$= \frac{\partial u_v}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u_v}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial u_w}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u_w}{\partial w}\frac{\partial w}{\partial x}$$

$$= \frac{\partial^2 u}{\partial v^2} + \frac{\partial^2 u}{\partial w \partial v} + \frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2}$$

$$= \frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2}$$

EXERCISE 3 : SHOW THAT $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2}\right)$

THUS, THE WAVE EQUATION BECOMES

$$\frac{\partial^2 u}{\partial v \partial w} = 0$$

INTEGRATE WITH RESPECT TO v TO GET

$$\frac{\partial u}{\partial w} = h(w) \qquad (h(w) \text{ ARBITRARY})$$

INTEGRATE WITH RESPECT TO w TO GET

$$u = \int h(w)\,dw + \phi(v)$$

BUT $\int h(w)\,dw$ IS AN ARBITRARY FUNCTION $\psi(w)$ OF w SO

$$u = \phi(v) + \psi(w).$$

$$\boxed{u(x,t) = \phi(x+ct) + \psi(x-ct)}$$

<u>EXERCISE</u> 4 :

1. SOLVE $\quad x\dfrac{\partial^2 u}{\partial x \partial y} = y\dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial u}{\partial y}$ BY MAKING THE CHANGE OF VARIABLE $v = x$, $w = xy$.

2. SOLVE $\quad \dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2}$ BY MAKING THE CHANGE OF VARIABLE $v = x$, $w = x - y$.

THIS PROCEDURE SIMPLE AND ELEGANT, BUT NOT PARTICULARLY USEFUL IN PRACTICAL SITUATIONS.

WE WILL NOW DESCRIBE (THE FIRST STEPS OF) A TECHNIQUE THAT IS THE BASIS FOR ALMOST ALL PRACTICAL CALCULATIONS.

HERE IS THE GENERAL PROBLEM WE WOULD LIKE TO SOLVE :

$$\begin{cases} c^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2} \quad , \quad 0 < x < \ell,\ t > 0 & \text{(1-DIMENSIONAL WAVE EQN)} \\ u(0,t) = u(\ell,t) = 0 \quad , \quad t \geq 0 & \text{(ENDPOINTS ARE FIXED)} \\ u(x,0) = f(x) \quad , \quad 0 \leq x \leq \ell & \text{(INITIAL DISPLACEMENT } f(x)\text{)} \\ \dfrac{\partial u}{\partial t}(x,0) = 0 \quad , \quad 0 \leq x \leq \ell & \text{(INITIAL VELOCITY ZERO)} \end{cases}$$

NOTE : FOR THE TIME BEING, THE INITIAL DISPLACEMENT OF THE STRING CAN BE ANY FUNCTION $f(x)$ ON $[0, \ell]$ WITH $f(0) = f(\ell) = 0$.

BEGIN BY LOOKING FOR SOLUTIONS TO THE FIRST TWO PARTS OF THE PROBLEM

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$u(0,t) = u(\ell,t) = 0$$

AND WORRY ABOUT THE LAST TWO PARTS LATER.

SEPARATION OF VARIABLES : AS A GUESS ("ANSATZ") WE WILL LOOK FOR SOLUTIONS OF THE FORM

$$u(x,t) = X(x)\,T(t)$$

MOTIVATION: $\frac{\partial u}{\partial x} = X'(x)T(t)$ AND $\frac{\partial u}{\partial t} = X(x)T'(t)$ SO THE PARTIAL DERIVATIVES ARE NOW ORDINARY DERIVATIVES AND WE HAVE SOME HOPE OF TURNING THE PDE (ABOUT WHICH WE KNOW NOTHING) INTO SOME ODEs (ABOUT WHICH WE KNOW A LITTLE).

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$c^2 X''(x)T(t) = X(x)T''(t)$$

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)}$$

THE LEFT-HAND SIDE IS A FUNCTION OF x AND THE RIGHT-HAND SIDE IS A FUNCTION OF t. THE ONLY WAY THEY CAN BE EQUAL IS IF EACH IS SOME CONSTANT σ SO

$$X''(x) - \sigma X(x) = 0$$

$$T''(t) - \sigma c^2 T(t) = 0$$

EACH OF THESE IS A CONSTANT COEFFICIENT EQUATION AND SO EASY TO SOLVE. HOWEVER, $u(0,t) = u(\ell,t) = 0$ IMPLIES

$$X(0)T(t) = X(\ell)T(t) = 0$$

FOR ALL t SO

$$X(0) = X(\ell) = 0$$

(OTHERWISE, $T(t) \equiv 0$ SO $u(x,t) \equiv 0$ WHICH IS A SOLUTION TO THE WAVE EQUATION, BUT BORING).

FIRST NOTE THAT IF

$$\begin{cases} X'' - \sigma X = 0 \\ X(0) = X(\ell) = 0 \end{cases}$$

IS TO HAVE NONZERO SOLUTIONS, THEN σ MUST BE NEGATIVE.

I. SUPPOSE $\sigma = 0$. THEN $X'' = 0$ SO $X(x) = k_1 x + k_2$
SO $X(0) = 0 \Rightarrow k_2 = 0$ AND $X(x) = k_1 x$. BUT
THEN $X(\ell) = 0 \Rightarrow k_1 \ell = 0 \Rightarrow k_1 = 0$ SO
$X(x) = 0$

II. SUPPOSE $\sigma > 0$, SAY, $\sigma = \mu^2$ FOR SOME $\mu > 0$. THEN
$X'' - \mu^2 X = 0 \Rightarrow X(x) = k_1 e^{\mu x} + k_2 e^{-\mu x}$

(CHARACTERISTIC EQUATION IS $r^2 - \mu^2 = 0$ SO $r = \pm\mu$)

$X(0) = 0 \Rightarrow k_1 + k_2 = 0$ SO $k_2 = -k_1$ AND
$X(x) = k_1 (e^{\mu x} - e^{-\mu x})$. THEN $X(\ell) = 0 \Rightarrow$

$$k_1 (e^{\mu \ell} - e^{-\mu \ell}) = 0$$

$$k_1 (e^{2\mu \ell} - 1) = 0$$

SINCE $\mu \neq 0$ AND $\ell \neq 0$ THIS IMPLIES $k_1 = 0$ SO
$k_2 = -k_1 = 0$ AND, AGAIN, $X(x) = 0$.

THUS, σ MUST BE NEGATIVE AND WE WILL WRITE IT AS

$$\sigma = -\lambda^2$$

FOR SOME REAL NUMBER $\lambda > 0$.

THUS, OUR EQUATIONS BECOME

$$X'' + \lambda^2 X = 0 \qquad (X(0) = X(\ell) = 0)$$

$$T'' + c^2\lambda^2 T = 0$$

EXERCISE 5 : SHOW THAT THE SOLUTION TO THE FIRST EQUATION IS $X(x) = A\cos\lambda x + B\sin\lambda x$ AND THAT $X(0) = 0 \Rightarrow A = 0$ SO

$$X(x) = B\sin\lambda x$$

NOW, $X(\ell) = 0$ IMPLIES $B\sin\lambda\ell = 0$. SINCE WE CAN'T ALLOW B TO BE ZERO (FOR THEN $X(x) \equiv 0$ AND $u(x,t) = 0$) WE MUST HAVE

$$\sin\lambda\ell = 0$$

I.E., $\lambda\ell = n\pi$ FOR SOME $n = 1, 2, \ldots$.

THUS, THE ONLY PERMISSIBLE VALUES OF λ ARE THE NUMBERS

$$\lambda_n = \frac{n\pi}{\ell}, \quad n = 1, 2, \ldots$$

THESE ARE CALLED THE EIGENVALUES FOR OUR PROBLEM. FOR EACH OF THESE WE OBTAIN A SOLUTION

$$X_n(x) = \sin\lambda_n x = \sin\frac{n\pi x}{\ell}$$

TO $X'' + \lambda_n^2 X = 0$ WITH $X_n(0) = X_n(\ell) = 0$.

FOR THESE VALUES OF λ THE T-EQUATION BECOMES

$$T'' + \frac{n^2\pi^2c^2}{\ell^2} T = 0$$

<u>EXERCISE</u> 6 : SHOW THAT, FOR EACH $n = 1, 2, \ldots$, THE GENERAL SOLUTION IS

$$T_n(t) = A_n \cos \frac{n\pi ct}{\ell} + B_n \sin \frac{n\pi ct}{\ell}$$

WHERE A_n AND B_n ARE ARBITRARY CONSTANTS.

THUS, FOR EACH $n = 1, 2, \ldots$ WE CAN BUILD SOLUTIONS $u_n(x,t) = X_n(x) T_n(t)$ TO THE WAVE EQUATION SATISFYING $u_n(0,t) = u_n(\ell,t) = 0$:

$$u_n(x,t) = \sin \frac{n\pi x}{\ell} \left(A_n \cos \frac{n\pi ct}{\ell} + B_n \sin \frac{n\pi ct}{\ell} \right)$$

<u>EXERCISE</u> 7 : SHOW THAT $\frac{\partial u_n}{\partial t}(x,0) = 0$ (INITIAL VELOCITY ZERO) REQUIRES THAT $B_n = 0$ SO

$$\boxed{u_n(x,t) = A_n \sin \frac{n\pi x}{\ell} \cos \frac{n\pi ct}{\ell}}$$

THESE ARE CALLED THE <u>EIGENFUNCTIONS</u> OF OUR PROBLEM AND THEY ALL SATISFY 3 OF THE 4 REQUIRED CONDITIONS (LACKING ONLY $u(x,0) = f(x)$).

EACH OF THESE FUNCTIONS REPRESENTS A POSSIBLE MOTION OF THE STRING. LET'S SEE WHAT INITIAL DISPLACEMENTS GIVE RISE TO SUCH MOTIONS (I.E., FOR WHICH $f(x)$ WE HAVE NOW SOLVED OUR PROBLEM COMPLETELY). FOR SIMPLICITY, WE WILL TAKE $A_n = 1$.

$$u_n(x,t) = \sin\frac{n\pi x}{\ell}\cos\frac{n\pi c t}{\ell}$$

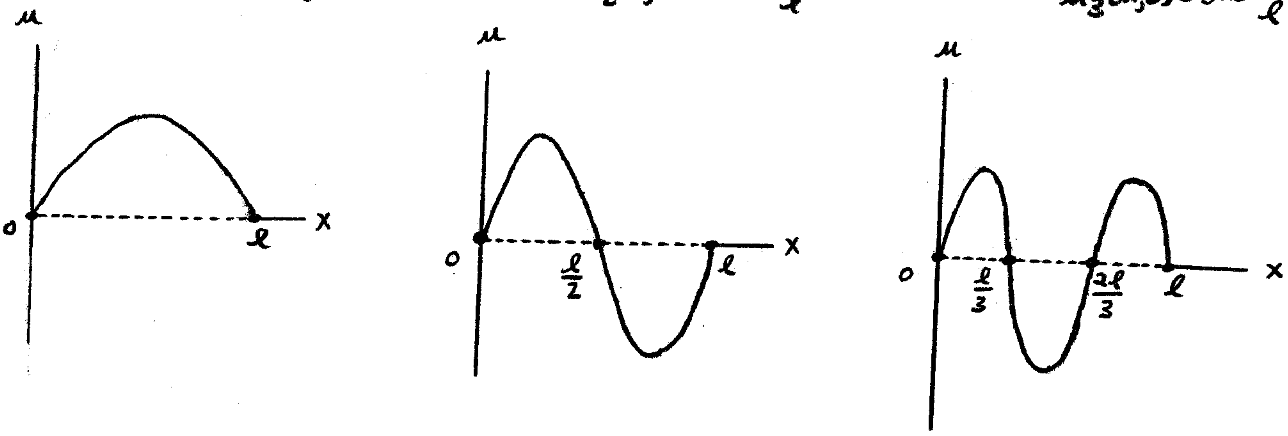
$$\Rightarrow \qquad u_n(x,0) = \sin\frac{n\pi x}{\ell}$$

SKETCH A FEW OF THESE INITIAL DISPLACEMENTS: $t=0$

$n=1$ $\qquad u_1(x,0) = \sin\frac{\pi x}{\ell}$

$n=2$ $\qquad u_2(x,0) = \sin\frac{2\pi x}{\ell}$

$n=3$ $\qquad u_3(x,0) = \sin\frac{3\pi x}{\ell}$



NOTE: FOR EACH $t_0 > 0$ THE DISPLACEMENT $u_n(x, t_0)$ IS JUST A MULTIPLE (BY $\cos\frac{n\pi c t_0}{\ell}$) OF THE INITIAL DISPLACEMENT. THESE ARE STANDING WAVES.

THE $u_n(x,t)$ ARE CALLED THE NORMAL MODES OF VIBRATION OF THE STRING. $n=1$ IS THE FIRST HARMONIC (OR FUNDAMENTAL MODE), WHILE $n>1$ ARE THE HIGHER HARMONICS (OR OVERTONES).

EXERCISE 8 : FIND THE SOLUTION TO THE VIBRATING STRING PROBLEM

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad , 0 < x < \ell ,\ t > 0$$

$$u(0,t) = u(\ell, t) = 0 , \qquad t \geq 0$$

$$u(x,0) = \sin \frac{2\pi x}{\ell} , \quad 0 \leq x \leq \ell$$

$$\frac{\partial u}{\partial t}(x,0) = 0 \qquad , \quad 0 \leq x \leq \ell$$

AT THIS POINT WE CAN ONLY SOLVE THE VIBRATING STRING PROBLEM WHEN THE INITIAL DISPLACEMENT HAPPENS TO BE OF THE FORM

$$f(x) = \sin \frac{k\pi x}{\ell} \qquad , 0 \leq x \leq \ell$$

AND THIS ISN'T MUCH.

HERE IS SOME HELP :

SUPERPOSITION PRINCIPLE : SUPPOSE $u_1(x,t), \ldots, u_n(x,t)$ ALL SATISFY

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$u(0,t) = u(\ell, t) = 0$$

$$\frac{\partial u}{\partial t}(x,0) = 0$$

AND $A_1, \ldots, A_n$ ARE REAL NUMBERS, THEN

$$u(x,t) = A_1 u_1(x,t) + \cdots + A_n u_n(x,t) = \sum_{k=1}^{n} A_k u_k(x,t)$$

ALSO SATISFIES ALL THREE.

THE REASON IS SIMPLE :

$$
\begin{aligned}
c^2 \frac{\partial^2 u}{\partial x^2} &= c^2 \frac{\partial^2}{\partial x^2} (A_1 u_1 + \cdots + A_n u_n) \\
&= c^2 \left(A_1 \frac{\partial^2 u_1}{\partial x^2} + \cdots + A_n \frac{\partial^2 u_n}{\partial x^2} \right) \\
&= A_1 \left(c^2 \frac{\partial^2 u_1}{\partial x^2} \right) + \cdots + A_n \left(c^2 \frac{\partial^2 u_n}{\partial x^2} \right) \\
&= A_1 \frac{\partial^2 u_1}{\partial t^2} + \cdots + A_n \frac{\partial^2 u_n}{\partial t^2} \\
&= \frac{\partial^2}{\partial t^2} (A_1 u_1 + \cdots + A_n u_n) \\
&= \frac{\partial^2 u}{\partial t^2}
\end{aligned}
$$

$$u(0,t) = A_1 u_1(0,t) + \cdots + A_n u_n(0,t) = A_1 \cdot 0 + \cdots + A_n \cdot 0 = 0$$

$$u(\ell,t) = A_1 u_1(\ell,t) + \cdots + A_n u_n(\ell,t) = A_1 \cdot 0 + \cdots + A_n \cdot 0 = 0$$

$$\frac{\partial u}{\partial t}(x,0) = A_1 \frac{\partial u_1}{\partial t}(x,0) + \cdots + A_n \frac{\partial u_n}{\partial t}(x,0) = A_1 \cdot 0 + \cdots + A_n \cdot 0 = 0$$

NOW NOTICE THAT THE FUNCTION

$$
\begin{aligned}
u(x,t) &= \sum_{k=1}^{n} A_k u_k(x,t) \\
&= \sum_{k=1}^{n} A_k \sin \frac{k\pi x}{\ell} \cos \frac{k\pi c t}{\ell}
\end{aligned}
$$

DESCRIBES THE MOTION OF THE STRING IF IT HAS INITIAL DISPLACEMENT

$$u(x,0) = \sum_{k=1}^{n} A_k \sin \frac{k\pi x}{\ell}$$

SO WE CAN NOW SOLVE THE VIBRATING STRING PROBLEM COMPLETELY WHENEVER THE INITIAL DISPLACEMENT IS OF THE FORM

$$f(x) = A_1 \sin \frac{\pi x}{\ell} + A_2 \sin \frac{2\pi x}{\ell} + \cdots + A_n \sin \frac{n\pi x}{\ell} .$$

EXERCISE 9 : SOLVE THE VIBRATING STRING PROBLEM

$$4 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} , \qquad 0 < x < 5, \ t > 0$$

$$u(0,t) = u(5,t) = 0 , \qquad t \geq 0$$

$$u(x,0) = \frac{12\pi}{5} \sin \frac{3\pi x}{5} + 12 \sin \frac{6\pi x}{5} , \quad 0 \leq x \leq 5$$

$$\frac{\partial u}{\partial t}(x,0) = 0 , \qquad 0 \leq x \leq 5$$

THE NEXT STEP (WHICH WE WILL NOT TAKE BECAUSE IT REQUIRES A FAIR AMOUNT OF WORK) IS TO REPLACE THE FINITE LINEAR COMBINATION OF EIGENFUNCTIONS

$$\sum_{k=1}^{n} A_k \sin \frac{k\pi x}{\ell} \cos \frac{k\pi c t}{\ell}$$

BY AN INFINITE SERIES

$$u(x,t) = \sum_{k=1}^{\infty} A_k \sin \frac{k\pi x}{\ell} \cos \frac{k\pi c t}{\ell} .$$

ASSUMING THAT ALL OF THE TECHNICAL ISSUES (CONVERGENCE, ETC.) CAN BE DEALT WITH (THEY CAN) THIS DESCRIBES THE

MOTION OF THE STRING IF THE INITIAL DISPLACEMENT IS

$$u(x,0) = \sum_{k=1}^{\infty} A_k \sin \frac{k\pi x}{\ell} .$$

AND NOW FOR THE SURPRISE. IT IS SHOWN IN THE THEORY OF <u>FOURIER SERIES</u> THAT VIRTUALLY ANY "REASONABLE" FUNCTION $f(x)$ CAN BE WRITTEN IN THE FORM

$$f(x) = \sum_{k=1}^{\infty} A_k \sin \frac{k\pi x}{\ell}$$

SO THE VIBRATING STRING PROBLEM IS BASICALLY SOLVED FOR ANY INITIAL DISPLACEMENT.

<u>EXERCISE</u> 10: A THIN STRAIGHT WIRE LIES ALONG THE INTERVAL $0 \le x \le \ell$ OF THE x-AXIS. THE INITIAL TEMPERATURE DISTRIBUTION IN THE WIRE IS GIVEN BY THE FUNCTION $f(x)$, $0 \le x \le \ell$. THE ENDS OF THE WIRE ARE HELD AT TEMPERATURE 0. IF $u(x,t)$ IS THE TEMPERATURE OF THE WIRE AT LOCATION x AND TIME t, THEN

$$\begin{cases} c^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t} & , 0 < x < \ell ,\ t > 0 \\ u(0,t) = u(\ell,t) = 0, & t \geq 0 \\ u(x,0) = f(x) & , 0 < x < \ell \end{cases}$$

WHERE c^2 IS THE "THERMAL DIFFUSIVITY" OF THE WIRE (A CONSTANT)

(a) SEPARATE VARIABLES AND SHOW THAT THE ONLY POSSIBLE VALUES OF THE SEPARATION CONSTANT GIVING NONTRIVIAL SOLUTIONS ARE

$$-\frac{k^2\pi^2}{\ell^2}, \quad k = 1, 2, \ldots$$

(b) SHOW THAT ALL OF THE FUNCTIONS

$$u_k(x,t) = e^{-k^2\pi^2c^2t/\ell^2} \sin\frac{k\pi x}{\ell},$$

$$k = 1, 2, \ldots$$

SATISFY THE FIRST TWO PARTS OF THE PROBLEM AND DETERMINE WHICH INITIAL TEMPERATURE GIVES RISE TO THE TEMPERATURE EVOLUTION $u_k(x,t)$.

(c) PROVE A SUPERPOSITION PRINCIPLE FOR THE HEAT CONDUCTION PROBLEM AND DESCRIBE ALL OF THE INITIAL TEMPERATURE DISTRIBUTIONS $f(x)$ FOR WHICH THE PROBLEM CAN BE SOLVED BY THESE SUPERPOSITIONS.

EXERCISE 11: DESCRIBE AT LEAST TWO PHYSICAL SITUATIONS GOVERNED BY

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 & , 0 < x < a, \ 0 < y < b \\ u(0,y) = u(a,y) = 0 & , \quad 0 \le y \le b \\ u(x,b) = 0 & , 0 \le x \le a \\ u(x,0) = f(x) & , 0 \le x \le a \end{cases}$$