

L'HÔPITAL'S RULE

RECALL : INDETERMINATE FORMS :

$$\frac{0}{0}$$

$$\lim \frac{f(x)}{g(x)} \rightarrow 0$$

$$\frac{\infty}{\infty}$$

$$\lim \frac{f(x)}{g(x)} \rightarrow \pm \infty$$

"EXTRA WORK REQUIRED" TO DETERMINE THE LIMIT. SOMETIMES THIS IS EASY (E.G., $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$) AND SOMETIMES IT'S NOT (E.G., $\lim_{x \rightarrow 0} \frac{\sin x}{x}$).

THIS SECTION INTRODUCES A NEW TOOL FOR COMPUTING SUCH LIMITS. TO SEE WHERE IT COMES FROM, THINK ABOUT THE FOLLOWING SITUATION :

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \rightarrow 0$$

WHERE $f(x)$ AND $g(x)$ ARE DIFFERENTIABLE (AND THEREFORE CONTINUOUS SO $f(a) = \lim_{x \rightarrow a} f(x) = 0$ AND $g(a) = \lim_{x \rightarrow a} g(x) = 0$).

THEN

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{g(x) - g(a)}$$

$$= \lim_{x \rightarrow a} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}}$$

$$= \frac{\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}}{\lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a}}$$

(PROVIDED THE DENOMINATOR IS NOT ZERO)

$$= \frac{f'(a)}{g'(a)}$$

$$= \frac{\lim_{x \rightarrow a} f'(x)}{\lim_{x \rightarrow a} g'(x)}$$

(PROVIDED $f'(x)$ AND $g'(x)$ ARE ALSO CONTINUOUS)

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

$$\text{E.G., } \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x^2 - 4)'}{(x - 2)'} = \lim_{x \rightarrow 2} \frac{2x}{1} = 2 \cdot 2 = 4$$

MORE CAREFUL ARGUMENTS SHOW THAT $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ FOR

ESSENTIALLY ALL OF THE VARIOUS TYPES OF LIMITS WE HAVE

COMPUTED ($x \rightarrow a$, $x \rightarrow a^+$, $x \rightarrow a^-$, $x \rightarrow \infty$, $x \rightarrow -\infty$) PROVIDED

THAT $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ IS OF TYPE $\frac{0}{0}$ OR $\frac{\infty}{\infty}$. THIS IS CALLED

L'HÔPITAL'S RULE : IF EITHER

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \rightarrow \frac{0}{0} \quad \text{OR} \quad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} \rightarrow \frac{\pm \infty}{\pm \infty}$$

THEN

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

(INCLUDING THE CASE IN WHICH $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \pm \infty$)

NOTE : THE ORIGINAL LIMIT MUST BE
INDETERMINATE OF TYPE $\frac{0}{0}$ OR $\frac{\infty}{\infty}$, E.G.,

$$\lim_{x \rightarrow 0} \frac{x+6}{x+2} = 3 \quad \text{AND THIS IS NOT THE
SAME AS} \quad \lim_{x \rightarrow 0} \frac{(x+6)'}{(x+2)'} = \lim_{x \rightarrow 0} \frac{1}{1} = 1.$$

EXAMPLES :

$$1. \quad \lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 5x} \rightarrow \frac{0}{0} = \lim_{x \rightarrow 0} \frac{2 \cos 2x}{5 \cos 5x} = \frac{2 \cdot 1}{5 \cdot 1} = \frac{2}{5}$$

$$2. \quad \lim_{x \rightarrow \infty} \frac{e^{3x}}{x} \rightarrow \frac{\infty}{\infty} = \lim_{x \rightarrow \infty} \frac{3e^{3x}}{1} = \infty$$

$$3. \quad \lim_{x \rightarrow 0} \frac{e^x - 1}{x^3} \rightarrow \frac{0}{0} = \lim_{x \rightarrow 0} \frac{e^x}{3x^2} \rightarrow \frac{1}{0} \quad \text{DNE}$$

$$e^x > 0 \quad \text{AND} \quad 3x^2 > 0 \Rightarrow$$

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x^3} = \infty$$

$$4. \lim_{x \rightarrow 0^-} \frac{\tan x}{x^2} \begin{matrix} \rightarrow 0 \\ \rightarrow 0 \end{matrix} = \lim_{x \rightarrow 0^-} \frac{\sec^2 x}{2x} \begin{matrix} \rightarrow 1 \\ \rightarrow 0 \end{matrix} \quad \text{DNE}$$

$2x \rightarrow 0$ THROUGH NEGATIVE VALUES SO

$$\lim_{x \rightarrow 0^-} \frac{\tan x}{x^2} = -\infty$$

$$5. \lim_{x \rightarrow 0^+} \frac{\ln x}{\csc x} \begin{matrix} \rightarrow -\infty \\ \rightarrow \infty \end{matrix} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\csc x \cot x} =$$

$$\lim_{x \rightarrow 0^+} - \frac{\sin^2 x}{x \cos x} \begin{matrix} \rightarrow 0 \\ \rightarrow 0 \end{matrix} = \lim_{x \rightarrow 0^+} - \frac{2 \sin x \cos x}{-x \sin x + \cos x}$$

$$= - \frac{2(0)(1)}{0 + 1} = - \frac{0}{1} = 0$$

OTHER TYPES OF INDETERMINATE FORMS :

$$0 \cdot \infty \quad \lim f(x)g(x)$$

$$\begin{matrix} \downarrow & \downarrow \\ 0 & \pm \infty \end{matrix}$$

$$\infty - \infty \quad \lim (f(x) - g(x))$$

$$\begin{matrix} \downarrow & \downarrow \\ \infty & \infty \end{matrix}$$

$$0^0 \quad \lim f(x)^{g(x)} \rightarrow 0$$

$$\downarrow$$

$$0$$

$$\infty^0 \quad \lim f(x)^{g(x) \rightarrow 0}$$

$$\downarrow$$

$$\pm \infty$$

$$1^\infty \quad \lim f(x)^{g(x) \rightarrow \infty}$$

$$\downarrow$$

$$1$$

THE FIRST TWO TYPES ARE HANDLED BY ALGEBRAICALLY TURNING THE PRODUCT / DIFFERENCE INTO A QUOTIENT AND USING L'HÔPITAL'S RULE.

EXAMPLES :

$$6. \quad \lim_{x \rightarrow 0^+} x \ln x$$

$$\downarrow \quad \downarrow$$

$$0 \quad -\infty$$

$$= \lim_{x \rightarrow 0^+} \frac{\ln x \rightarrow -\infty}{\frac{1}{x} \rightarrow \infty} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} -x = 0$$

$$7. \quad \lim_{x \rightarrow \frac{\pi}{4}} (1 - \tan x) \sec 2x$$

$$\downarrow \quad \downarrow$$

$$0 \quad \infty$$

$$= \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x \rightarrow 0}{\cos 2x \rightarrow 0} = \lim_{x \rightarrow \frac{\pi}{4}} \frac{-\sec^2 x}{-2 \sin 2x} = \frac{\sec^2 \frac{\pi}{4}}{2 \sin \frac{\pi}{2}} = \frac{2}{2 \cdot 1} = 1$$

$$8. \lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \infty & & \infty \end{array}$$

$$= \lim_{x \rightarrow 0^+} \left(\frac{1}{x} \frac{\sin x}{\sin x} - \frac{1}{\sin x} \frac{x}{x} \right) = \lim_{x \rightarrow 0^+} \frac{\sin x - x}{x \sin x} \begin{array}{l} \rightarrow 0 \\ \rightarrow 0 \end{array}$$

$$= \lim_{x \rightarrow 0^+} \frac{\cos x - 1}{x \cos x + \sin x} \begin{array}{l} \rightarrow 0 \\ \rightarrow 0 \end{array} = \lim_{x \rightarrow 0^+} \frac{-\sin x}{-x \sin x + \cos x + \cos x}$$

$$= \frac{-0}{0+1+1} = \frac{0}{2} = 0$$

NOTE: THERE'S NOTHING INDETERMINATE ABOUT

$$\lim_{x \rightarrow 0^-} \left(\frac{1}{x^2} - \frac{1}{\sin x} \right) = \infty$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \infty & & -\infty \end{array}$$

THE "EXPONENTIAL TYPE" INDETERMINATE FORMS ARE HANDLED BY
TAKING LOGARITHMS, EVALUATING THE LIMIT OF THE LOGARITHM, AND
THEN EXPONENTIATING TO GET THE ORIGINAL LIMIT.

EXAMPLES:

$$9. \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x \rightarrow \infty$$

$$\downarrow$$

$$1$$

LET $y = \left(1 + \frac{1}{x}\right)^x$ (WE WANT TO FIND $\lim_{x \rightarrow \infty} y$)

THEN

$$\ln y = \ln \left(1 + \frac{1}{x}\right)^x$$

$$\ln y = x \ln \left(1 + \frac{1}{x}\right)$$

$$\ln y = \frac{\ln \left(1 + \frac{1}{x}\right)}{\frac{1}{x}}$$

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{x}\right)}{\frac{1}{x}} \rightarrow \frac{0}{0}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{1 + \frac{1}{x}} \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{1 + \frac{1}{x}} = \frac{1}{1 + 0} = 1$$

THUS,

$$\lim_{x \rightarrow \infty} \ln y = 1$$

SO

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} e^{\ln y} = e^1 = e$$

$$10. \quad \lim_{x \rightarrow 0^+} x^x \rightarrow 0$$

$$\quad \quad \quad \downarrow$$

$$\quad \quad \quad 0$$

$$\text{LET } y = x^x$$

$$\ln y = \ln(x^x)$$

$$\ln y = x \ln x$$

$$\ln y = \frac{\ln x}{\frac{1}{x}}$$

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \frac{\ln x \rightarrow -\infty}{\frac{1}{x} \rightarrow \infty}$$

$$= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} -x = 0$$

so

$$\lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} e^{\ln y} = e^0 = 1$$