LOCALIZATION AND STATIONARY PHASE APPROXIMATION

TOPOLOGY WHERE YOU LEAST EXPECT IT

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1. THE LOCALIZATION PHENOMENON

2. STATIONARY PHASE APPROXIMATION

3. HAMILTONIAN ACTIONS ON SYMPLECTIC MANIFOLDS

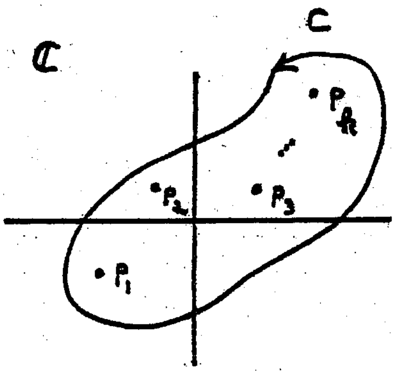
4. EQUIVARIANT COHOMOLOGY AND LOCALIZATION

5. TOPOLOGICAL QUANTUM FIELD THEORY

1. THE LOCALIZATION PHENOMENON

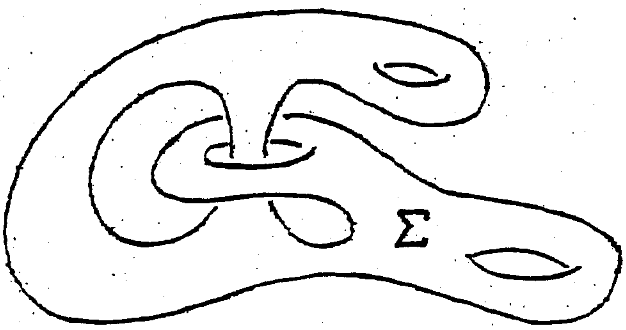
E.G.,

RESIDUE THEOREM



$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \mathrm{Res}(f, p_j)$$

GAUSS-BONNET + POINCARÉ-HOPF



$\chi(\Sigma)$ = EULER CHARACTERISTIC OF Σ =

$$\frac{1}{2\pi}\int_{\Sigma} K = \sum_{\substack{p\in\Sigma \\ V(p)=0}} \mathrm{Index}\,(V(p))$$

2. STATIONARY PHASE APPROXIMATION

ASYMPTOTIC ($t \to \infty$) BEHAVIOR OF INTEGRALS SUCH AS

$$\int_X e^{itH(x)} dx$$

(OR $t = i\beta$ AND $\beta \to \infty$)

NO HARM WILL BEFALL YOU (FOR A LITTLE WHILE) IF YOU THINK OF X AS A COMPACT SURFACE (SPHERE, TORUS, ...) AND THE INTEGRAL AS AN ORDINARY SURFACE INTEGRAL.

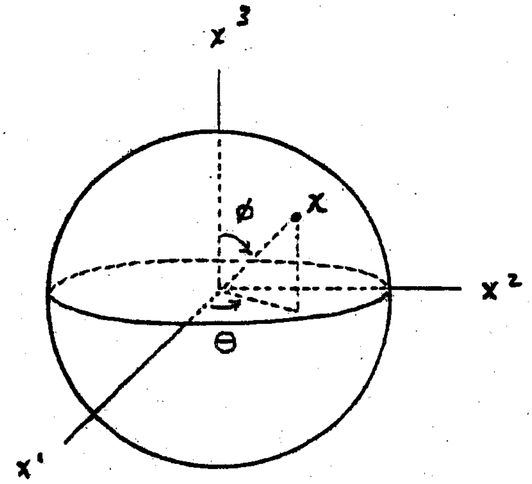
FOR THE FUTURE, HOWEVER, HERE'S THE TECHNICAL STUFF :

X = COMPACT, ORIENTED, SMOOTH MANIFOLD OF DIMENSION $n = 2k$

dx = VOLUME FORM ON X

H = A MORSE FUNCTION ON X
(SMOOTH, REAL-VALUED FUNCTION WITH ALL CRITICAL POINTS ($dH(p) = 0$) NONDEGENERATE (NONSINGULAR HESSIAN $\mathcal{H}_H(p)$))

EXAMPLE :



$$S^2 = \{ (x^1, x^2, x^3) \in \mathbb{R}^3 : (x^1)^2 + (x^2)^2 + (x^3)^2 = 1 \}$$

H = HEIGHT FUNCTION ON S^2 : $H(x) = x^3 = \cos\phi$

$$\int_{S^2} e^{itH(x)}\,dx = \int_{S^2} e^{it\cos\phi} \sin\phi \, d\phi \, d\theta = 4\pi \left(\frac{\sin t}{t} \right)$$

GENERALLY, SUCH INTEGRALS ARE NOT SO EASY TO EVALUATE AND ONE MUST SETTLE FOR APPROXIMATIONS.

ROUGHLY, THE "STATIONARY PHASE APPROXIMATION" AMOUNTS TO THE ASSERTION THAT, FOR LARGE t, THE DOMINANT CONTRIBUTIONS TO SUCH AN INTEGRAL COME FROM NEIGHBORHOODS OF THE CRITICAL POINTS OF H.

$$\int_X e^{itH(x)}\,dx = \boxed{\sum_{dH(p)=0} \left(\frac{2\pi}{t}\right)^{k} \frac{e^{\pi i(\mathrm{SGN}\,\mathcal{H}_H(p))/4}\, e^{itH(p)}}{|\det \mathcal{H}_H(p)|^{\frac{1}{2}}}} + O(t^{-(k+1)})$$

THE $\boxed{\text{SUM}}$ IS THE STATIONARY PHASE APPROXIMATION.

NOT AS WEIRD AS IT LOOKS: ON $\mathbb{R}$ CONSIDER $\int_{-\infty}^{\infty} e^{itH(x)}dx$ AND SUPPOSE $H(x)$ HAS ONE CRITICAL POINT p ($H'(p)=0$) THAT'S NONDEGENERATE ($H''(p)\neq 0$). APPROXIMATE

$$H(x) \approx H(p) + \tfrac{1}{2}H''(p)(x-p)^2$$

TO GET

$$e^{itH(p)}\int_{-\infty}^{\infty} e^{itH''(p)(x-p)^2/2}\,dx.$$

LOOK UP THE GAUSSIAN INTEGRAL AND OBTAIN

$$\sqrt{\frac{2\pi}{t}}\;\frac{e^{\pi i(\mathrm{SIGN}\,H''(p))/4}\;e^{itH(p)}}{|H''(p)|^{\frac{1}{2}}}$$

<u>EXAMPLE</u> : $\displaystyle\int_{S^2} e^{itH(x)}dx$, H = HEIGHT FUNCTION ON S^2

CRITICAL POINTS OF H : $N = (0,0,1)$ AND $S = (0,0,-1)$

$$\mathcal{H}_H(N) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \text{SGN}\, \mathcal{H}_H(N) = -2$$

$$\mathcal{H}_H(S) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \text{SGN}\, \mathcal{H}_H(S) = 2$$

$$\left(\frac{2\pi}{t}\right)^1 \frac{e^{\pi i(\text{SGN}\,\mathcal{H}_H(N))/4}\, e^{itH(N)}}{|\det \mathcal{H}_H(N)|^{\frac{1}{2}}} + \left(\frac{2\pi}{t}\right)^1 \frac{e^{\pi i(\text{SGN}\,\mathcal{H}_H(S))/4}\, e^{itH(S)}}{|\det \mathcal{H}_H(S)|^{\frac{1}{2}}}$$

$$= \left(\frac{2\pi}{t}\right) e^{-\pi i/2} e^{it} + \left(\frac{2\pi}{t}\right) e^{\pi i/2} e^{-it}$$

$$= \frac{2\pi i}{t}\left(e^{-it} - e^{it}\right) = 4\pi\left(\frac{\sin t}{t}\right)$$

WHICH IS PRECISELY THE VALUE OF THE INTEGRAL (SEE PAGE 3).

> FOR THE HEIGHT FUNCTION ON S^2,
> THE STATIONARY PHASE APPROXIMATION
> IS EXACT.

THIS IS <u>NOT</u> THE CASE FOR THE HEIGHT FUNCTION ON THE TORUS (OR ANY OTHER COMPACT SURFACE OF POSITIVE GENUS).

3. HAMILTONIAN ACTIONS ON SYMPLECTIC MANIFOLDS

MORE JARGON (EXAMPLE SHORTLY):

A SYMPLECTIC FORM ON X IS A NONDEGENERATE, CLOSED 2-FORM σ ON X (THINK OF IT AS A SORT OF "SKEW-SYMMETRIC INNER PRODUCT" ON TANGENT VECTORS WITH $d\sigma = 0$). σ DETERMINES A CANONICAL VOLUME FORM

$$dx_\sigma = \frac{1}{k!}\, \overbrace{\sigma \wedge \cdots \wedge \sigma}^{k}$$

ON X CALLED THE LIOUVILLE FORM.

$$\boxed{\begin{array}{c}\text{ON } S^2 \text{ (OR ANY COMPACT SURFACE)} \\ dx = \sigma = dx_\sigma\end{array}}$$

ON A SYMPLECTIC MANIFOLD (X, σ) ANY REAL-VALUED FUNCTION H DETERMINES A HAMILTONIAN VECTOR FIELD V_H

$$dH(\cdot) = \sigma(V_H, \cdot) \qquad (= \iota_{V_H}\sigma)$$

THE INTEGRAL CURVES OF V_H THEN DETERMINE A HAMILTONIAN FLOW ON X. H IS CONSTANT ALONG THE INTEGRAL CURVES OF V_H.

> PHYSICISTS SHOULD THINK OF (X, σ) AS A CLASSICAL PHASE SPACE, H AS A HAMILTONIAN, AND V_H AS THE VECTOR FIELD DETERMINED BY HAMILTON'S EQUATIONS.

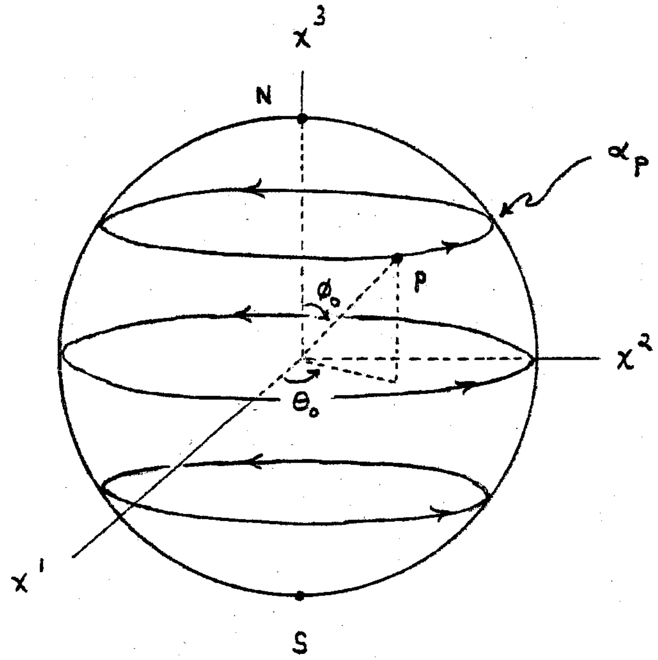
FOR THE HEIGHT FUNCTION H ON S^2 THE HAMILTONIAN VECTOR FIELD IS

$$V_H = \partial_\theta$$

(TAKEN TO BE ZERO AT N AND S).

THE UNIQUE INTEGRAL CURVE THROUGH p (SPHERICAL COORDINATES (ϕ_0, θ_0)) AT $t = 0$ IS

$$\alpha_p(t) = \left(\sin\phi_0 \cos(\theta_0 + t), \sin\phi_0 \sin(\theta_0 + t), \cos\phi_0\right)$$



THE FLOW

$$\alpha : S^2 \times \mathbb{R} \longrightarrow S^2$$

$$\alpha(p, t) = \alpha_p(t)$$

IS <u>PERIODIC</u> (NONTRIVIAL ORBITS HAVE THE SAME MINIMAL PERIOD).

DUISTERMAAT - HECKMAN THEOREM :

(X, σ) , $\dim X = 2k$

$H \in C^\infty(X)$ MORSE

HAMILTONIAN VECTOR FIELD V_H HAS PERIODIC FLOW

$\Rightarrow$

$$\int_X e^{itH} dx_\sigma = \sum_{\substack{p \in X \\ dH(p)=0}} \left(\frac{2\pi}{t}\right)^k \frac{e^{\pi i (\operatorname{sgn} \mathcal{H}_H(p))/4} \, e^{itH(p)}}{|\det \mathcal{H}_H(p)|^{\frac{1}{2}}}$$

$\forall t > 0$.

NOW LET'S MOVE ONE LAYER DEEPER. ANOTHER WAY TO VIEW A PERIODIC FLOW :

GENERATES AN S^1-ACTION ON X :

$$(e^{it}, p) \longrightarrow e^{it} \cdot p = \alpha_p(t)$$

FOCUS ON GROUP ACTIONS :

(X, σ) , $\dim X = 2k$

G A COMPACT LIE GROUP (LIE ALGEBRA $\mathfrak{g}$)

G - ACTION ON X :

$$(g, p) \longrightarrow g \cdot p$$

INFINITESIMAL ACTION OF $\mathfrak{g}$: $\xi \in \mathfrak{g} \longrightarrow$ VECTOR FIELD $\xi^{\#}$

$$\xi^{\#}(p) = \frac{d}{dt}(\exp(-t\xi)\cdot p)\Big|_{t=0}$$

$$Z(\xi^{\#}) = \text{ZERO SET OF } \xi^{\#}$$

ASSUME THE ACTION IS <u>HAMILTONIAN</u> , I.E., THERE IS AN EQUIVARIANT MAP

$$\mu : \mathfrak{g} \longrightarrow C^{\infty}(X)$$

WITH

$$d\mu(\xi) = \iota_{\xi^{\#}}\sigma$$

$\forall \ \xi \in \mathfrak{g}$.

<u>NOTE</u> : NONDEGENERACY OF σ IMPLIES

$Z(\xi^{\#})$ = CRITICAL POINTS OF $\mu(\xi)$

FOR $G = S^1$ THIS COINCIDES WITH

THE FIXED POINT SET OF THE ACTION.

GENERALIZED DUISTERMAAT - HECKMAN THEOREM :

IF $Z(\xi^*)$ IS FINITE, THEN

$$\int_X e^{i\mu(\xi)} dx_\sigma = \sum_{p \in Z(\xi^*)} \frac{(2\pi i)^k e^{i\mu(\xi)(p)}}{Pf(L_p(\xi))}$$

WHERE $L_p(\xi) : T_p(X) \longrightarrow T_p(X)$ IS THE INFINITESIMAL ACTION ON $T_p(X)$ INDUCED BY ξ^* AND Pf IS THE PFAFFIAN.

IF $Z(\xi^*)$ IS NOT FINITE, IT IS A SUBMANIFOLD OF X (PERHAPS CONSISTING OF SEVERAL COMPONENTS OF DIFFERENT DIMENSION) AND THE SUM ABOVE IS REPLACED BY A SUM OVER THE CONNECTED COMPONENTS WITH EACH TERM IN THE SUM BEING AN INTEGRAL INVOLVING THE "EQUIVARIANT EULER CLASS OF THE NORMAL BUNDLE OF $Z(\xi^*)$."

THIS IS NOT THE END, HOWEVER. IT WAS EVENTUALLY REALIZED THAT THE PROPER CONTEXT WITHIN WHICH TO VIEW THESE LOCALIZATION THEOREMS WAS (EQUIVARIANT) TOPOLOGY.

4. EQUIVARIANT COHOMOLOGY AND LOCALIZATION

RECALL THAT ANY MANIFOLD HAS A DE RHAM COMPLEX $\Omega^*(X)$:

$$\Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(X)$$

$$d^2 = 0$$

EXACT FORMS (IMAGE OF d) $\subseteq$ CLOSED FORMS (KERNEL OF d)

DE RHAM COHOMOLOGY :

$$H^p(X) = \text{CLOSED } p\text{-FORMS} \Big/ \text{EXACT } p\text{-FORMS}$$

NOW SUPPOSE THERE IS A G-ACTION ON X.

FREE ACTION $\Rightarrow$ ORBIT SPACE X/G IS A MANIFOLD AND SO HAS A DE RHAM COHOMOLOGY $H^*(X/G)$

CARTAN SHOWED HOW TO CALCULATE $H^*(X/G)$ FROM A COMPLEX <u>ON</u> <u>X</u> AND SO PROVIDED A REASONABLE ALTERNATIVE TO THE DE RHAM COHOMOLOGY OF X/G EVEN WHEN IT IS NOT A MANIFOLD.

CARTAN MODEL FOR THE <u>G-EQUIVARIANT</u> <u>COHOMOLOGY</u> OF X :

$$\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(X) \;=\; \text{SUMS OF } \alpha = \rho \otimes \varphi$$

$$= \Omega^*(X)\text{-VALUED POLYNOMIALS ON } \mathfrak{g}$$

$$\alpha(\xi) = (\rho \otimes \varphi)(\xi) = \rho(\xi)\varphi$$

INDUCED ACTION OF G ON $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(X)$:

$$(g \cdot \alpha)(\xi) = \rho(g^{-1}\xi g)\, L^*_{g^{-1}}(\varphi)$$

$$\Omega^*_G(X) = G\text{-INVARIANT ELEMENTS OF } \mathbb{C}[\mathfrak{g}] \otimes \Omega^*(X)$$

$$= \underline{G}\text{-}\underline{\text{EQUIVARIANT}}\ \underline{\text{DIFFERENTIAL}}\ \underline{\text{FORMS}} \text{ ON } X$$

<u>G-EQUIVARIANT</u> <u>EXTERIOR</u> <u>DERIVATIVE</u>

$$d_G : \Omega^*_G(X) \longrightarrow \Omega^*_G(X)$$

$$(d_G\alpha)(\xi) = d(\alpha(\xi)) - \iota_{\xi^\#}(\alpha(\xi)) = (d - \iota_{\xi^\#})(\alpha(\xi))$$

$(\Omega^*_G(X), d_G)$ IS A COCHAIN COMPLEX $(d_G^2 = 0)$ AND ITS COHOMOLOGY

$$H^*_G(X)$$

IS THE (CARTAN MODEL OF THE) G-EQUIVARIANT COHOMOLOGY OF X.

$$G\text{-ACTION FREE} \implies H^*_G(X) \cong H^*(X/G)$$

LIKE ORDINARY FORMS, G-EQUIVARIANT FORMS CAN BE INTEGRATED:

$$\int_X \; : \; \Omega^*_G(X) \to \mathbb{C}[\mathfrak{g}]$$

$$\left(\int_X \alpha \right)(\xi) \; = \; \int_X \alpha(\xi) \; := \; \int_X \alpha(\xi)_{[n]}$$

<u>EQUIVARIANT</u> <u>LOCALIZATION</u> <u>THEOREM</u>:

α G-EQUIVARIANTLY CLOSED ($d_G \alpha = 0$) $\Rightarrow$

FOR ANY $\xi \in \mathfrak{g}$ FOR WHICH $Z(\xi^\#)$ IS FINITE

$$\int_X \alpha(\xi) \; = \; \sum_{p \in Z(\xi^\#)} (-2\pi)^k \, \frac{\alpha(\xi)_{[0]}(p)}{\mathrm{Pf}(L_p(\xi))}$$

IF $Z(\xi^\#)$ IS NOT FINITE THE SUM IS OVER ITS CONNECTED COMPONENTS AND THE TERMS ARE INTEGRALS INVOLVING THE EQUIVARIANT EULER CLASS OF THE NORMAL BUNDLE OF $Z(\xi^\#)$.

ENTER : THE PHYSICISTS

5. TOPOLOGICAL QUANTUM FIELD THEORY

QUANTUM FIELD THEORY IS OFTEN PHRASED IN TERMS OF "PATH INTEGRALS" OVER "INFINITE-DIMENSIONAL MANIFOLDS."

REGRETTABLY, THESE INTEGRALS GENERALLY MAKE NO MATHEMATICAL SENSE AT ALL (NO PROBLEM FOR THE PHYSICIST).

ED WITTEN HAD THE IDEA OF "FORMALLY" APPLYING THE LOCALIZATION THEOREMS TO CERTAIN PHASE SPACE PATH INTEGRALS WHERE THE PHYSICS "SUGGESTS" A STRUCTURE "ANALOGOUS TO" THE HYPOTHESES OF THE FINITE-DIMENSIONAL THEOREMS.

THE "RESULTS" WERE SPECTACULAR, E.G.,

A. LOOP SPACE AND THE ATIYAH-SINGER INDEX THEOREM

B. 2-DIMENSIONAL YANG-MILLS THEORY AND THE MODULI SPACE OF FLAT CONNECTIONS ON A RIEMANN SURFACE

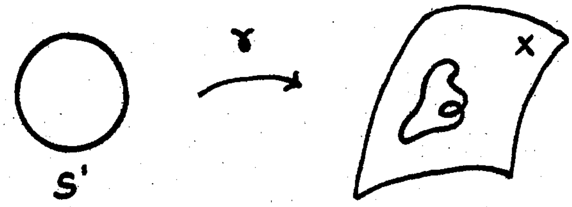
C. 4-DIMENSIONAL YANG-MILLS THEORY AND THE DONALDSON INVARIANTS

TIME PERMITTING, A BRIEF SKETCH OF

LOOP SPACE AND THE ATIYAH-SINGER INDEX THEOREM

(X, σ) + RIEMANNIAN METRIC + SPIN STRUCTURE

LOOP SPACE : $C^\infty(S^1, X)$



$T_\gamma(C^\infty(S^1, X))$ = VECTOR FIELDS ALONG γ

SYMPLECTIC STRUCTURE : $\hat{\sigma}_\gamma(V_1, V_2) = \int_0^1 \sigma_{\gamma(t)}(V_1(\gamma(t)), V_2(\gamma(t)))\,dt$

NATURAL S^1-ACTION ON $C^\infty(S^1, X)$ ("ROTATE LOOPS") IS HAMILTONIAN :

$$H : C^\infty(S^1, X) \longrightarrow \mathbb{R}$$

$$H(\gamma) = \text{"ENERGY" OF } \gamma = \tfrac{1}{2}\int_0^1 \|\gamma'(t)\|^2\,dt$$

FIXED POINT SET = CONSTANT LOOPS $\cong X$

DUISTERMAAT-HECKMAN FOR

$$\int_{C^\infty(S^1, X)} e^{itH(\gamma)}\,dx_{\hat{\sigma}}$$

(PATH INTEGRAL REPRESENTATION FOR HEAT KERNEL OF DIRAC OPERATOR ON X)

GIVES

$$\int_X \hat{A}(X)$$

(ATIYAH-SINGER THEOREM FOR THE INDEX OF THE DIRAC OPERATOR ON X)

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