I WOULD LIKE TO MENTION ONE OTHER ITEM RELATED TO GEODESICS THAT WE WILL NOT HAVE TIME TO PROVE (IT TAKES A FAIR AMOUNT OF PRETTY SERIOUS ANALYSIS), BUT WHICH IS AN EXTREMELY USEFUL TOOL.

GIVEN M WITH A CONNECTION ∇, $p \in M$ AND $v \in T_p(M)$, LET α_v BE THE UNIQUE MAXIMAL GEODESIC THAT FITS v AT p:

$$\alpha_v(0) = p$$

$$\alpha_v'(0) = v$$

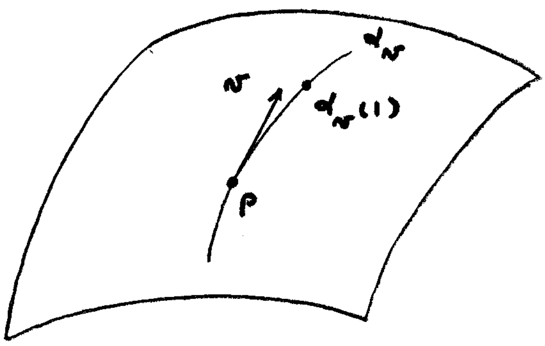
THIS IS DEFINED ON SOME INTERVAL ABOUT 0 THAT WILL GENERALLY BE DIFFERENT FOR DIFFERENT $v \in T_p(M)$.

CONSIDER THE MAP

$$T_p(M) \longrightarrow M$$

$$v \longrightarrow \alpha_v(1)$$

DEFINED ON THE SET OF TANGENT VECTORS AT p FOR WHICH THE DOMAIN OF α_v INCLUDES 1 (WHICH INCLUDES AT LEAST THE ZERO ELEMENT OF $T_p(M)$).



THEOREMS FROM DIFFERENTIAL EQUATIONS GUARANTEE THAT THIS MAP IS, IN FACT, SMOOTH ON SOME OPEN NEIGHBORHOOD OF $0 \in T_p(M)$.

MOREOVER, ONE CAN COMPUTE ITS DERIVATIVE AS A MAP FROM THE MANIFOLD $T_p(M)$ TO THE MANIFOLD M AND ONE FINDS THAT IT IS NONSINGULAR AT $0 \in T_p(M)$.

THE INVERSE FUNCTION THEOREM THEN IMPLIES THAT THIS MAP, DENOTED

$$\exp_p(v) = \alpha_v(1)$$

IS A <u>DIFFEOMORPHISM</u> ON SOME OPEN NEIGHBORHOOD OF $0 \in T_p(M)$, I.E., IT IS THE INVERSE OF A <u>CHART</u> ON M AT p.

IT'S NOT HARD TO SEE THAT

$$\exp_p(tv) = \alpha_v(t)$$

FOR $0 \le t \le 1$ SO ONE CAN ASSUME THAT THE OPEN SET U_0 IN $T_p(M)$ ON WHICH $\exp_p$ IS A DIFFEOMORPHISM IS <u>STAR-SHAPED</u> (I.E., $v \in U_0 \Rightarrow tv \in U_0$ FOR $0 \le t \le 1$).

THE IMAGE $\exp_p(U_0)$ IS AN OPEN COORDINATE NEIGHBORHOOD OF p IN M AND ANY SUCH IS CALLED A <u>NORMAL NEIGHBORHOOD</u> OF p.

PICKING A BASIS FOR $T_p(M)$ IDENTIFIES IT WITH $\mathbb{R}^n$ AND THEN $\exp_p^{-1}$ SUPPLIES COORDINATES NEAR p IN M (CALLED <u>NORMAL COORDINATES</u>)

NOTE THAT ANY POINT IN A NORMAL NEIGHBORHOOD OF p CAN BE JOINED TO p BY A GEODESIC OF M, ∇ CONTAINED IN THAT NORMAL NEIGHBORHOOD. WITH A GREAT DEAL MORE EFFORT ONE CAN PROVE THE

<u>THEOREM</u> : LET M BE A SMOOTH MANIFOLD WITH A CONNECTION ∇. THEN EACH $p \in M$ HAS A NORMAL NEIGHBORHOOD U_p WHICH IS A NORMAL NEIGHBORHOOD OF EACH OF ITS POINTS.

IN PARTICULAR, ANY TWO POINTS IN U_p CAN BE JOINED BY A GEODESIC LYING IN U_p.

IF YOU WOULD LIKE TO SEE A PROOF OF ALL OF THIS, HERE'S A REFERENCE :

DIFFERENTIAL GEOMETRY, LIE GROUPS, AND SYMMETRIC SPACES, SIGURDUR HELGASON, ACADEMIC PRESS, 1978, PAGES 32 - 36.

NOW ITS TIME TO START LOOKING AT SOME CONCRETE EXAMPLES. WE WILL ISOLATE THE EXAMPLES OF INTEREST BY SHOWING THAT IF M HAPPENS TO HAVE A RIEMANNIAN METRIC g, THEN g UNIQUELY DETERMINES A CONNECTION ON M THAT IS NATURALLY "ADAPTED" TO IT.

MOTIVATION:

RECALL THAT THE STANDARD CONNECTION ON $\mathbb{R}^n$, GIVEN RELATIVE TO STANDARD COORDINATES $x^1, \ldots, x^n$ BY

$$\begin{aligned} \nabla_V W &= \nabla_V \left(W^j \frac{\partial}{\partial x^j} \right) \\ &= V(W^j) \frac{\partial}{\partial x^j} \end{aligned}$$

HAS TWO PROPERTIES THAT ARE GENERALLY NOT SHARED BY ARBITRARY CONNECTIONS:

1. $\nabla_V W - \nabla_W V = [V, W]$

2. $X(V \cdot W) = V \cdot \nabla_X W + \nabla_X V \cdot W$ (THIS ONE DOESN'T MAKE SENSE IN GENERAL UNLESS M HAS A RIEMANNIAN METRIC.)

MOREOVER, FROM THESE TWO IT FOLLOWS PURELY ALGEBRAICALLY THAT

3. $$\begin{aligned} \nabla_V W \cdot X &= V(W \cdot X) - X(V \cdot W) + W(X \cdot V) \\ &\quad + W \cdot [X, V] - V \cdot [W, X] + X \cdot [V, W]. \end{aligned}$$

ALTHOUGH IT MAY SEEM AT THE MOMENT THAT THESE MAY BE VERY SPECIAL PROPERTIES OF A VERY SPECIAL CONNECTION, I WOULD LIKE YOU TO CONVINCE YOURSELF THAT A GREAT MANY "NATURAL" CONNECTIONS ON RIEMANNIAN MANIFOLDS SHARE THEM.

<u>EXERCISE</u> : LET M BE A SUBMANIFOLD OF $\mathbb{R}^n$. GIVE M THE INDUCED RIEMANNIAN METRIC g ($g = \iota^* \tilde{g}$, WHERE $\tilde{g}$ IS THE STANDARD RIEMANNIAN METRIC ON $\mathbb{R}^n$ AND $\iota : M \hookrightarrow \mathbb{R}^n$ IS THE INCLUSION MAP). LET V AND W BE VECTOR FIELDS ON M. DEFINE $\nabla_V W$ AS FOLLOWS : FOR EACH $p \in M$ THERE IS AN OPEN NEIGHBORHOOD $\tilde{U}$ OF p IN $\mathbb{R}^n$ AND VECTOR FIELDS $\tilde{V}, \tilde{W}$ ON $\tilde{U}$ WHOSE RESTRICTIONS TO $M \cap \tilde{U}$ AGREE WITH V, W (WHY?). IF $\tilde{\nabla}$ IS THE STANDARD CONNECTION ON $\mathbb{R}^n$, THEN $\tilde{\nabla}_{\tilde{V}} \tilde{W}(p)$ IS GENERALLY NOT TANGENT TO M SO TAKE

$$\nabla_V W(p) = \left(\tilde{\nabla}_{\tilde{V}} \tilde{W}(p) \right)^T$$

WHERE " T " MEANS THE ORTHOGONAL PROJECTION INTO $T_p(M)$. SHOW THAT ∇ IS A CONNECTION ON M AND THAT IT, LIKE $\tilde{\nabla}$, SATISFIES PROPERTIES # 1 AND 2.

A CONNECTION ∇ ON M IS SAID TO BE SYMMETRIC IF

$$\nabla_V W - \nabla_W V = [V, W]$$

$\forall$ $V, W \in \mathcal{T}(TM)$.

REASON: THEN

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$$

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}$$

SO

$$\Gamma^k_{ij} = \Gamma^k_{ji} \quad , k = 1, \ldots, n .$$

NOTE: FAILURE TO SATISFY THIS CONDITION IS MEASURED BY THE TORSION

$$T(V, W) = \nabla_V W - \nabla_W V - [V, W].$$

SYMMETRY HAS NOTHING TO DO WITH RIEMANNIAN STRUCTURE, BUT RATHER RELATES THE CONNECTION ∇ TO THE UNDERLYING MANIFOLD STRUCTURE VIA $[\ ,\ ]$.

NOW SUPPOSE M HAS DEFINED ON IT BOTH A CONNECTION ∇ AND A RIEMANNIAN METRIC g.

> NOTE: IN RELATIVITY THE METRIC IS "SEMI-RIEMANNIAN" BUT EVERYTHING WE SAY FOR THE REMAINDER OF THIS DISCUSSION APPLIES EQUALLY WELL TO THESE.

GENERALLY, ∇ AND g HAVE NOTHING TO DO WITH EACH OTHER, BUT WE WOULD LIKE TO CONSIDER ONLY THE CASE IN WHICH THEY ARE NATURALLY RELATED IN THE FOLLOWING SENSE:

> PARALLEL TRANSLATION ALONG A SMOOTH CURVE α, AS DETERMINED BY ∇, SHOULD PRESERVE "DOT PRODUCTS", AS DETERMINED BY g.

WE INTEND TO PROVE THAT THIS IS EQUIVALENT TO THE CONDITION THAT THE RATE OF CHANGE OF THE DOT PRODUCT $g(V, W)$ OF TWO VECTOR FIELDS WITH RESPECT TO A THIRD VECTOR FIELD X CAN BE COMPUTED FROM THE PRODUCT RULE

$$X(g(V,W)) = g(V, \nabla_X W) + g(\nabla_X V, W)$$

(WHEN THIS CONDITION IS SATISFIED WE SAY THAT THE CONNECTION ∇ IS <u>COMPATIBLE</u> WITH THE METRIC g).

<u>THEOREM</u>: LET M BE A SMOOTH MANIFOLD ON WHICH IS DEFINED A CONNECTION ∇ AND A RIEMANNIAN METRIC g. THEN THE FOLLOWING ARE EQUIVALENT.

1. FOR ALL VECTOR FIELDS V, W AND X ON M,

$$X(g(V,W)) = g(V, \nabla_X W) + g(\nabla_X V, W)$$

2. FOR ANY SMOOTH CURVE $\alpha : I \to M$ AND ANY TWO VECTOR FIELDS X AND Y ALONG α,

$$\frac{d}{dt} g(X,Y) = g(X, \frac{DY}{dt}) + g(\frac{DX}{dt}, Y)$$

3. FOR ANY SMOOTH CURVE $\alpha : I \to M$ AND ANY TWO PARALLEL VECTOR FIELDS P AND P' ALONG α,

$$g(P, P') = \text{CONSTANT}$$

ON I.

<u>PROOF</u>: THE MOST OBVIOUS PART OF THIS IS THAT #2 $\Rightarrow$ #3 (BECAUSE $\frac{DP}{dt} = 0$ AND $\frac{DP'}{dt} = 0$). LET'S HANDLE THE CONVERSE OF THIS FIRST. SO, WE ASSUME THAT PARALLEL TRANSLATION PRESERVES DOT PRODUCTS AND TRY TO DERIVE THE FORMULA IN #2.

FIX SOME $t_0 \in I$. CHOOSE A g-ORTHONORMAL BASIS

$$\{ P_1(t_0), \ldots, P_n(t_0) \}$$

FOR $T_{\alpha(t_0)}(M)$. PARALLEL TRANSLATE EACH VECTOR $P_i(t_0)$ ALONG α TO OBTAIN $P_i(t) \in T_\alpha(TM)$. BY ASSUMPTION,

$$\{ P_1(t), \ldots, P_n(t) \}$$

IS A g-ORTHONORMAL BASIS FOR $T_{\alpha(t)}(M)$ FOR EACH $t \in I$. THUS, GIVEN TWO VECTOR FIELDS X AND Y ALONG α WE CAN WRITE

$$X(t) = X^i(t) P_i(t) \quad , \quad Y(t) = Y^j(t) P_j(t)$$

$\forall\, t \in I$. MOREOVER,

$$\frac{DX}{dt} = \frac{D}{dt}(X^i P_i) = X^i \frac{DP_i}{dt} + \frac{dX^i}{dt} P_i = \frac{dX^i}{dt} P_i$$

AND, SIMILARLY,

$$\frac{DY}{dt} = \frac{dY^j}{dt} P_j$$

(THESE ARE WORTH REMEMBERING - RELATIVE TO A PARALLEL TRANSLATED BASIS, COVARIANT DIFFERENTIATION ALONG A CURVE IS JUST COMPONENTWISE DIFFERENTIATION WITH RESPECT TO t.)

SINCE $\{P_1(t), \dots, P_n(t)\}$ IS g-ORTHONORMAL,

$$g(X,Y) = \sum_{i=1}^{n} x^i y^i$$

$$g\left(X, \frac{DY}{dt}\right) = \sum_{i=1}^{n} x^i \frac{dy^i}{dt}$$

$$g\left(\frac{DX}{dt}, Y\right) = \sum_{i=1}^{n} \frac{dx^i}{dt} y^i$$

SO

$$\begin{aligned} g\left(X, \frac{DY}{dt}\right) + g\left(\frac{DX}{dt}, Y\right) &= \sum_{i=1}^{n} \left(x^i \frac{dy^i}{dt} + \frac{dx^i}{dt} y^i\right) \\ &= \frac{d}{dt} \sum_{i=1}^{n} x^i y^i \\ &= \frac{d}{dt} g(X,Y) \end{aligned}$$

AS REQUIRED.

THUS, WE HAVE SHOWN THAT #2 AND #3 ARE EQUIVALENT.

ALL THAT REMAINS IS TO SHOW THAT #1 AND #2 ARE EQUIVALENT.

NOTICE THAT BOTH OF THESE ASSERT THE EQUALITY OF TWO <u>FUNCTIONS</u> (OF $p \in M$ FOR #1 AND OF $t \in I$ FOR #2) AND SO CAN BE ESTABLISHED POINTWISE.

FIRST, LET US ASSUME #2 AND FIX A POINT $p \in M$. WE MUST SHOW THAT

$$X(p)(g(V,W)) = g_p(V(p), \nabla_{X(p)} W) + g_p(\nabla_{X(p)} V, W(p))$$

CHOOSE A SMOOTH CURVE $\alpha : I \to M$ WITH $\alpha(0) = p$ AND $\alpha'(0) = X(p)$. THEN

$$\begin{aligned}
X(p)(g(V,W)) &= \alpha'(0)(g(V,W)) \\
&= \frac{d}{dt}(g(V(\alpha(t)), W(\alpha(t))))\Big|_{t=0} \\
&= g_p\left(V(\alpha(0)), \frac{DW}{dt}(0)\right) + g_p\left(\frac{DV}{dt}(0), W(\alpha(0))\right) \\
&\qquad \text{BY \# 2} \\
&= g_p\left(V(p), \frac{DW}{dt}(0)\right) + g_p\left(\frac{DV}{dt}(0), W(p)\right) \\
&= g_p(V(p), \nabla_{\alpha'(0)} W) + g_p(\nabla_{\alpha'(0)} V, W(p)) \\
&\qquad \text{BY THE THIRD DEFINING PROPERTY OF } \frac{D}{dt} \\
&= g_p(V(p), \nabla_{X(p)} W) + g_p(\nabla_{X(p)} V, W(p))
\end{aligned}$$

AS REQUIRED.

<u>EXERCISE</u> : SHOW THAT $\# 1 \Rightarrow \# 2$. $\square$

NOW WE PROVE OUR MAJOR RESULT.

THEOREM: LET M BE A SMOOTH MANIFOLD WITH A RIEMANNIAN METRIC g. THEN THERE IS A UNIQUE CONNECTION ∇ ON M THAT SATISFIES

1. $\nabla_V W - \nabla_W V = [V,W] \quad \forall \; V,W \in \Gamma(TM)$

2. $X(g(V,W)) = g(V, \nabla_X W) + g(\nabla_X V, W)$

$$\forall \; X,V,W \in \Gamma(TM).$$

PROOF: ONCE AGAIN WE WILL PROVE UNIQUENESS FIRST AND THEN WORRY ABOUT EXISTENCE. SO, ASSUME THERE IS A CONNECTION ∇ ON M SATISFYING #1 AND #2. WE HAVE ALREADY SHOWN THAT IT FOLLOWS PURELY ALGEBRAICALLY FROM #1 AND #2 THAT, FOR ALL $V, W, X \in \Gamma(TM)$,

$$2g(\nabla_V W, X) = V(g(W,X)) - X(g(V,W)) + W(g(X,V)) + g(W,[X,V]) - g(V,[W,X]) + g(X,[V,W])$$

SINCE THE RIGHT-HAND SIDE IS COMPLETELY DETERMINED BY g, SO IS $g(\nabla_V W, X) \quad \forall \; V,W,X \in \Gamma(TM)$. REGARDING V AND W AS FIXED, THIS MEANS THAT THE DOT PRODUCTS OF $\nabla_V W$ WITH <u>EVERY</u>

X ARE COMPLETELY DETERMINED BY g AND THIS COMPLETELY DETERMINES $\nabla_V W$ (BY NONDEGENERACY OF g).

THUS, A CONNECTION ON M SATISFYING #1 AND #2 IS UNIQUELY DETERMINED BY THESE PROPERTIES.

TO PROVE EXISTENCE WE JUST TURN THE ARGUMENT AROUND, I.E., FOR ANY $V, W \in \mathcal{T}(TM)$ WE DEFINE $\nabla_V W$ BY SPECIFYING THAT, FOR ANY $X \in \mathcal{T}(TM)$,

$$g(\nabla_V W, X) = \frac{1}{2}\Big(V(g(W,X)) - X(g(V,W)) + W(g(X,V))$$
$$+ g(W,[X,V]) - g(V,[W,X]) + g(X,[V,W]) \Big)$$

(THE <u>KOSZUL FORMULA</u>)

THIS UNIQUELY DETERMINES $\nabla_V W$ BY NONDEGENERACY. ONE NEED ONLY VERIFY THAT $\nabla_V W$ SATISFIES THE DEFINING PROPERTIES OF A CONNECTION AS WELL AS #1, 2 OF THE THEOREM. I WILL PROVE ONE OF THESE AND LEAVE THE REST FOR YOU.

TO SHOW THAT $\nabla_V W - \nabla_W V = [V, W]$ IT WILL SUFFICE TO PROVE THAT, $\forall X \in \mathcal{T}(TM)$,

$$g(\nabla_V W - \nabla_W V, X) = g([V,W], X)$$

BUT

$$g(\nabla_V W - \nabla_W V, X) = g(\nabla_V W, X) - g(\nabla_W V, X)$$

$$= \tfrac{1}{2}\Big[\cancel{V(g(W,X))} - \cancel{X(g(V,W))} + \cancel{W(g(X,V))}$$

$$+ \cancel{g(W,[X,V])} - \cancel{g(V,[W,X])} + g(X,[V,W]) \Big]$$

$$- \tfrac{1}{2}\Big[\cancel{W(g(V,X))} - \cancel{X(g(W,V))} + \cancel{V(g(X,W))}$$

$$+ \cancel{g(V,[X,W])} - \cancel{g(W,[V,X])} + g(X,[W,V]) \Big]$$

$$= \tfrac{1}{2} g(X,[V,W]) - \tfrac{1}{2} g(X,[W,V])$$

$$= \tfrac{1}{2} g(X,[V,W]) + \tfrac{1}{2} g(X,[V,W])$$

$$= g([V,W], X)$$

<u>EXERCISE</u> : FINISH THE PROOF. $\square$

THIS UNIQUELY DETERMINED CONNECTION ON A RIEMANNIAN MANIFOLD (M, g) IS CALLED ITS <u>LEVI - CIVITA CONNECTION</u> .

HOW DOES ONE FIND IT ?

RECALL THAT ANY CONNECTION IS UNIQUELY DETERMINED BY ITS COLLECTION OF CHRISTOFFEL SYMBOLS FOR THE CHARTS IN SOME ATLAS. WE SHOW NOW THAT, FOR THE LEVI-CIVITA CONNECTION, THESE CAN BE COMPUTED FROM THE METRIC COMPONENTS.

LET (U, φ) BE A CHART WITH COORDINATE FUNCTIONS $x^1, \dots, x^n$. THEN Γ^k_{ij} ARE DEFINED BY

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}$$

USE THE KOSZUL FORMULA TO COMPUTE

$$g\left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^m}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x^i}\left(g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^m}\right)\right) - \frac{\partial}{\partial x^m}\left(g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\right) + \frac{\partial}{\partial x^j}\left(g\left(\frac{\partial}{\partial x^m}, \frac{\partial}{\partial x^i}\right)\right)\right)$$

(BECAUSE ALL BRACKETS ARE ZERO)

$$g\left(\Gamma^k_{ij} \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^m}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x^i}(g_{jm}) - \frac{\partial}{\partial x^m}(g_{ij}) + \frac{\partial}{\partial x^j}(g_{mi})\right)$$

$$\Gamma^k_{ij}\, g\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^m}\right) = \frac{1}{2}\left(\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m}\right)$$

$$g_{km}\, \Gamma^k_{ij} = \frac{1}{2}\left(\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m}\right)$$

$$g_{am}\, \Gamma^a_{ij} = \frac{1}{2}\left(\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m}\right)$$

MULTIPLY ON BOTH SIDES BY g^{km} AND SUM OVER m AS INDICATED TO GET

$$\underbrace{g^{km} g_{ma}}_{\delta^k_a} \Gamma^a_{ij} = \frac{1}{2} g^{km} \left(\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m} \right)$$

AND SO

$$\boxed{\Gamma^k_{ij} = \frac{1}{2} g^{km} \left(\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m} \right) \qquad i, j, k = 1, \dots, n}$$

NOTE : FOR STANDARD COORDINATES ON $\mathbb{R}^n$, ALL OF THE Γ^k_{ij} ARE ZERO

WE WILL COMPUTE SOME MORE INTERESTING EXAMPLES.