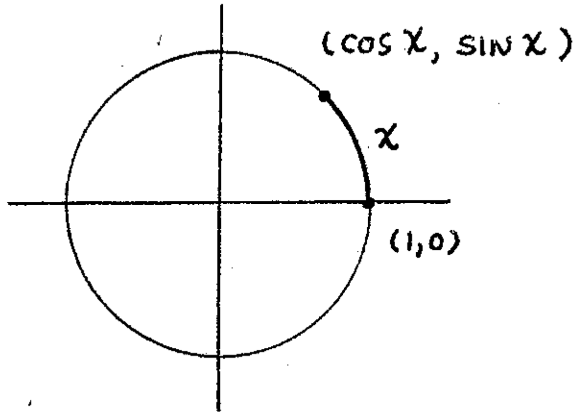
# LIMITS OF TRIG FUNCTIONS : THE SQUEEZE THEOREM




COS $x$ AND SIN $x$ ARE CONTINUOUS FOR ALL $x$

$\tan x$, $\cot x$, $\sec x$ AND $\csc x$ ARE CONTINUOUS FOR THOSE $x$ AT WHICH THE DENOMINATOR IS NONZERO, E.G.,

$\tan x = \dfrac{\sin x}{\cos x}$ IS CONTINUOUS FOR

ALL $x$ <u>EXCEPT</u> $\pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \ldots$

I.E.,

$$\frac{(2n+1)\pi}{2}, \quad n = 0, \pm 1, \pm 2, \ldots$$

SOME LIMITS :

$$\lim_{x \to 0} \cos x = \cos 0 = 1$$

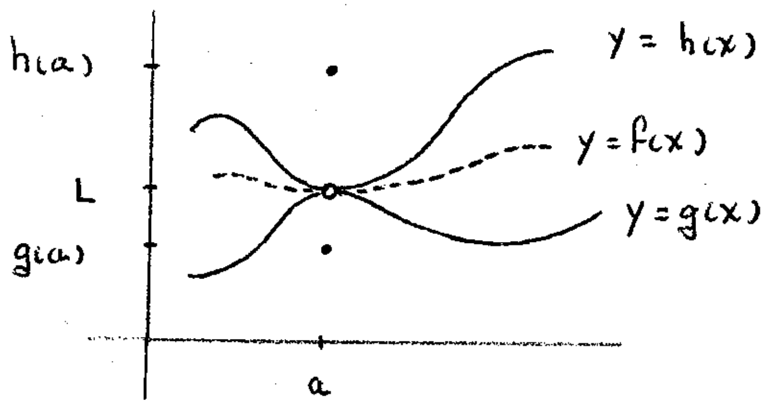
$$\lim_{x \to 0} \sin x = \sin 0 = 0$$

$$\lim_{x \to 1} \cos (x^2 - 1) = \cos \left( \lim_{x \to 1} (x^2 - 1) \right)$$

$$= \cos (1^2 - 1) = \cos 0 = 1$$

SOME LIMITS ARE NOT SO EASY AND REQUIRE A NEW TOOL :

THE SQUEEZE THEOREM :




$$\lim_{x \to a} g(x) = L$$

$$\lim_{x \to a} h(x) = L$$

$$g(x) \le f(x) \le h(x)$$

$$\Longrightarrow \quad \lim_{x \to a} f(x) = L \text{ ALSO}$$

EXAMPLE : $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x^2}\right)$

NOTE THAT, FOR ANY $x \neq 0$,

$$-1 \le \sin\left(\frac{1}{x^2}\right) \le 1$$

SO, MULTIPLYING BY $x^2$ (WHICH IS POSITIVE),

$$-x^2 \le x^2 \sin\left(\frac{1}{x^2}\right) \le x^2$$

SINCE $\lim_{x \to 0} (-x^2) = 0$ AND $\lim_{x \to 0} (x^2) = 0$ WE CONCLUDE THAT

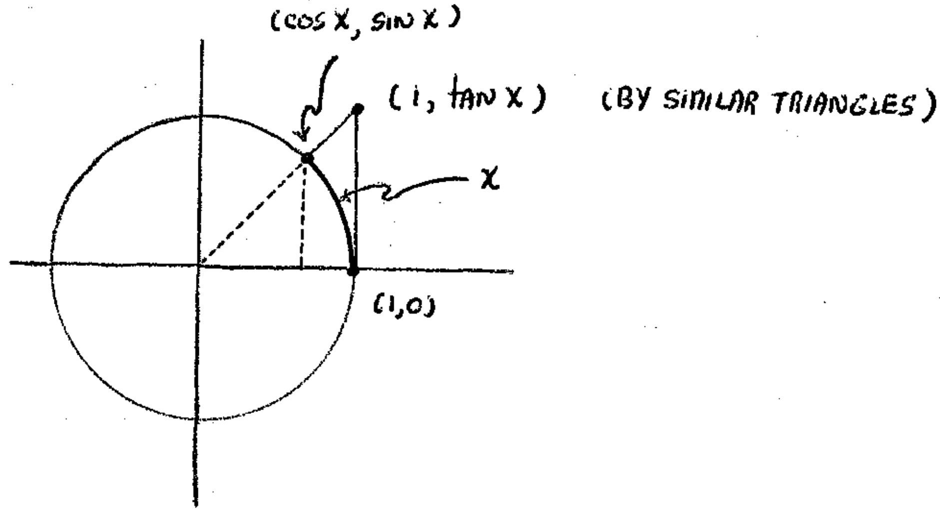
$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x^2}\right) = 0$$

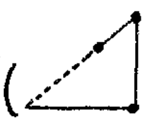
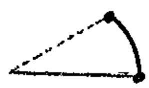
ALSO.

A MUCH MORE IMPORTANT EXAMPLE : WE WILL SHOW THAT

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

REMEMBER THIS ONE !




AREA (  ) $\geq$ AREA (  ) $\geq$ AREA (  )

$$\tfrac{1}{2}(1)\tan x \quad \geq \quad \tfrac{1}{2}(1^2)\,x \quad \geq \quad \tfrac{1}{2}\cos x \sin x$$

$$\tan x \quad \geq \quad x \quad \geq \quad \cos x \sin x$$

$$\frac{1}{\cos x} \quad \geq \quad \frac{x}{\sin x} \quad \geq \quad \cos x$$

$$\cos x \quad \leq \quad \frac{\sin x}{x} \quad \leq \quad \frac{1}{\cos x}$$

SINCE $\lim_{x \to 0} \cos x = \cos 0 = 1$ AND $\lim_{x \to 0} \frac{1}{\cos x} = \frac{1}{\cos 0} = 1$,

THE SQUEEE THEOREM IMPLIES THAT $\lim_{x \to 0} \frac{\sin x}{x} = 1$ ALSO.

SOME MORE LIMITS YOU CAN GET FROM THIS ONE :

1.

$$\boxed{\lim_{x \to 0} \frac{1 - \cos x}{x} = 0}$$

REMEMBER THIS ONE !

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} \frac{1 - \cos x}{x} \frac{1 + \cos x}{1 + \cos x} = \lim_{x \to 0} \frac{1 - \cos^2 x}{x(1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x(1 + \cos x)} = \lim_{x \to 0} \frac{\sin x}{x} \frac{\sin x}{1 + \cos x}$$

$$= (1)(0) = 0$$

2. $$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\frac{\sin x}{\cos x}}{x} = \lim_{x \to 0} \frac{\sin x}{x} \frac{1}{\cos x}$$

$$= (1)(1) = 1$$

3. $$\lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{x \to 0} 3 \left( \frac{\sin 3x}{3x} \right) = 3 \lim_{x \to 0} \frac{\sin 3x}{3x}$$

NOW LET $t = 3x$ AND NOTICE THAT

$x \to 0$ IF AND ONLY IF $t \to 0$

$$= 3 \lim_{t \to 0} \frac{\sin t}{t} = 3(1) = 3$$

4. $$\lim_{\theta \to 0} \left( \frac{\theta + \sin \theta}{\theta} \right) = \lim_{\theta \to 0} \left( 1 + \frac{\sin \theta}{\theta} \right) = 1 + 1 = 2$$

LOGARITHMIC AND EXPONENTIAL FUNCTIONS ARE ALSO CONTINUOUS ON THEIR DOMAINS, AS ARE THE INVERSE TRIGONOMETRIC FUNCTIONS.