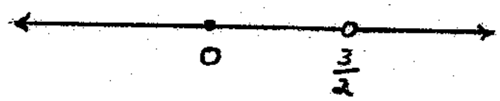
LIMITS

BEGIN BY THINKING ABOUT THE FUNCTION

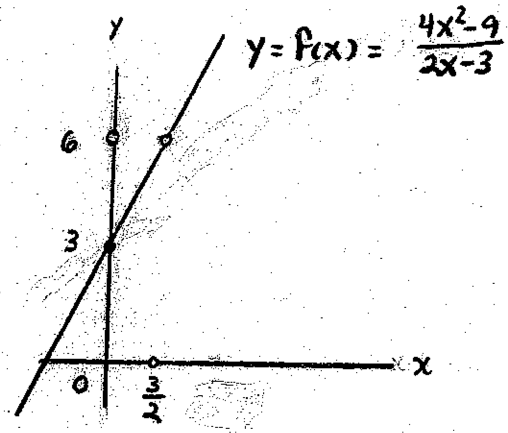
$$f(x) = \frac{4x^2-9}{2x-3}$$

DOMAIN : ALL $x \in \mathbb{R}$ EXCEPT $\frac{3}{2}$.



FOR $x \neq \frac{3}{2}$, $\frac{4x^2-9}{2x-3} = \frac{(2x-3)(2x+3)}{2x-3} = 2x+3$.

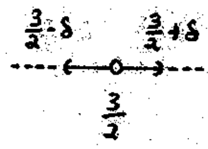
GRAPH OF f :



THE FUNCTION f IS NOT DEFINED AT $\frac{3}{2}$, BUT IT APPEARS TO "WANT" TO BE DEFINED AT $\frac{3}{2}$ AND TO TAKE THE VALUE 6 THERE (THE VALUES OF $f(x)$ CAN BE MADE AS CLOSE AS WE LIKE TO 6 SIMPLY BY CHOOSING x SUFFICIENTLY CLOSE, BUT NOT EQUAL TO $\frac{3}{2}$).

MORE PRECISELY, GIVEN ANY $\varepsilon > 0$ (HOWEVER SMALL)

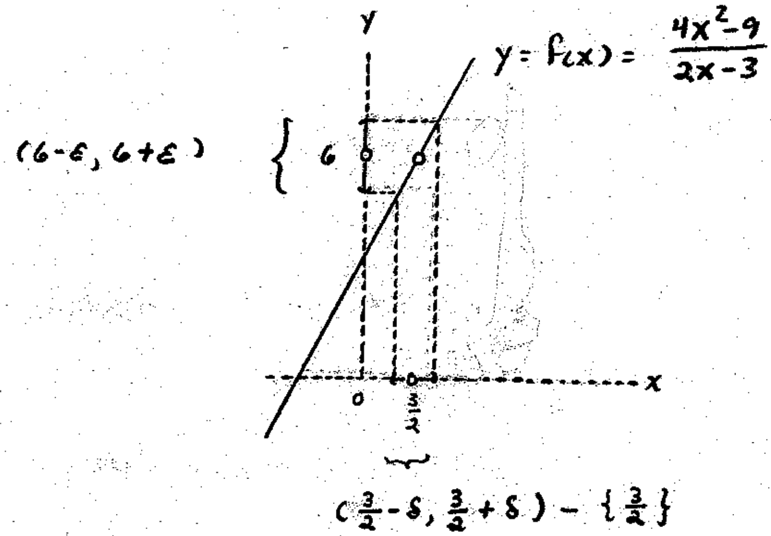
THERE IS A (SUFFICIENTLY SMALL) $\delta > 0$



SUCH THAT

$$0 < |x - \tfrac{3}{2}| < \delta \implies |f(x) - 6| < \varepsilon .$$

HERE'S THE PICTURE :



HERE'S THE ARITHMETIC : FOR $x \neq \frac{3}{2}$,

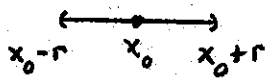
$$\begin{aligned} |f(x) - 6| &= \left| \frac{4x^2-9}{2x-3} - 6 \right| \\ &= |2x+3-6| \\ &= |2x-3| \\ &= 2\,|x - \tfrac{3}{2}| \end{aligned}$$

WHICH WILL BE $< \varepsilon$ IF $0 < |x - \frac{3}{2}| < \frac{\varepsilon}{2}$ SO WE CAN TAKE $\delta = \frac{\varepsilon}{2}$.

IN ORDER TO SAY THIS IN A NICE, GEOMETRICAL WAY WE INTRODUCE A LITTLE TERMINOLOGY AND NOTATION.

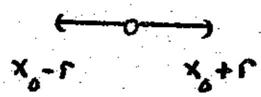
FIX AN $x_0 \in \mathbb{R}$ AND A POSITIVE REAL NUMBER r. DEFINE

$$U_r(x_0) = (x_0 - r, x_0 + r) = \underline{r}\text{-}\underline{\text{NEIGHBORHOOD}}\ \underline{\text{OF}}\ x_0$$



AND

$$U_r'(x_0) = U_r(x_0) - \{x_0\} = \underline{\text{DELETED}}\ \underline{r}\text{-}\underline{\text{NEIGHBORHOOD}}\ \underline{\text{OF}}\ x_0$$



WHAT WE HAVE SHOWN FOR $f(x) = \dfrac{4x^2-9}{2x-3}$ CAN THEN BE PHRASED THIS WAY :

FOR EVERY $\varepsilon > 0$ THERE EXISTS A $\delta > 0$ (NAMELY, $\frac{\varepsilon}{2}$)
SUCH THAT

$$f(U_\delta'(\tfrac{3}{2})) \subseteq U_\varepsilon(6).$$

CUTTING DOWN ON THE ENGLISH, WE WOULD WRITE

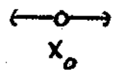
$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } f(U_\delta'(\tfrac{3}{2})) \subseteq U_\varepsilon(6).$$

ACCORDING TO THE DEFINITIONS WE ARE ABOUT TO WRITE DOWN, WHAT WE HAVE SHOWN IS THAT "THE LIMIT OF $\dfrac{4x^2-9}{2x-3}$ AS x APPROACHES $\frac{3}{2}$ IS 6", WRITTEN

$$\text{"} \lim_{x \to \frac{3}{2}} \frac{4x^2-9}{2x-3} = 6 \text{"}.$$

NOW, THE DEFINITIONS :

LET f BE A FUNCTION DEFINED (AT LEAST) ON SOME DELETED NEIGHBORHOOD OF $x_0 \in \mathbb{R}$.



IF L IS A REAL NUMBER WITH THE PROPERTY THAT

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \text{s.t.} \ 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

$$\text{(I.E., } f(U'_\delta(x_0)) \subseteq U_\varepsilon(L) \text{)}$$

THEN WE SAY THAT <u>THE LIMIT OF</u> $f(x)$ <u>AS</u> x <u>APPROACHES</u> x_0 <u>IS</u> L AND WRITE

$$\lim_{x \to x_0} f(x) = L$$

OR

$$f(x) \to L \quad \text{AS} \quad x \to x_0 .$$

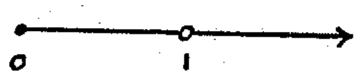
IF NO SUCH REAL NUMBER L EXISTS WE SAY THAT <u>THE LIMIT</u> $\lim_{x \to x_0} f(x)$ <u>DOES NOT EXIST</u>.

TO GET SOME FEEL FOR THE DEFINITIONS WE WILL WORK THROUGH A NUMBER OF

<u>EXAMPLES</u> :

1. $f(x) = \dfrac{\sqrt{x} - 1}{x - 1}$, $x_0 = 1$

   DOMAIN OF f : $\{x \in \mathbb{R} : x \geq 0 \text{ AND } x \neq 1\}$



THUS, f IS DEFINED ON SOME DELETED NBD (ABBREVIATION FOR "NEIGHBORHOOD") OF $x_0 = 1$ (E.G., $U_1'(1)$)

TO APPLY THE DEFINITION OF $\lim_{x \to x_0} f(x)$ WE NEED AN L, I.E., WE NEED TO HAVE ALREADY "GUESSED" THE ANSWER. IN THIS CASE WE NOTE THAT, FOR $x \neq 1$,

$$\frac{\sqrt{x}-1}{x-1} = \frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)} = \frac{1}{\sqrt{x}+1}$$

WHICH "SEEMS" TO APPROACH $\frac{1}{\sqrt{1}+1} = \frac{1}{2}$ AS $x \to 1$. NOW WE PROVE IT.

LET $\varepsilon > 0$ BE GIVEN.

<u>NOTE</u>: WE MUST FIND A $\delta > 0$ SUCH THAT

$$0 < |x-1| < \delta \implies \left| \frac{\sqrt{x}-1}{x-1} - \frac{1}{2} \right| < \varepsilon .$$

NOW, FOR $x \neq 1$,

$$\left| \frac{\sqrt{x}-1}{x-1} - \frac{1}{2} \right| = \left| \frac{1}{\sqrt{x}+1} - \frac{1}{2} \right| = \left| \frac{2-(\sqrt{x}+1)}{2(\sqrt{x}+1)} \right|$$

$$= \left| \frac{1-\sqrt{x}}{2(\sqrt{x}+1)} \right| = \left| \frac{\sqrt{x}-1}{2(\sqrt{x}+1)} \right|$$

$$= \left| \frac{\sqrt{x}-1}{2(\sqrt{x}+1)} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} \right| = \left| \frac{x-1}{2(\sqrt{x}+1)^2} \right|$$

$$= \frac{|x-1|}{2(\sqrt{x}+1)^2}$$

$$< |x-1| \quad \text{SINCE } 2(\sqrt{x}+1)^2 > 1 .$$

WHICH WILL BE $< \varepsilon$ IF $|x-1| < \varepsilon$. THUS, IF WE TAKE

$\delta = \varepsilon$ WE HAVE THAT

$$0 < |x-1| < \delta = \varepsilon \Rightarrow \left| \frac{\sqrt{x}-1}{x-1} - \frac{1}{2} \right| < |x-1| < \varepsilon$$

AS REQUIRED SO

$$\lim_{x \to 1} \frac{\sqrt{x}-1}{x-1} = \frac{1}{2} .$$

2. $f(x) = x^4$, $x_0 = 2$

DOMAIN : $\mathbb{R}$

WE SHOW THAT $\lim_{x \to 2} x^4 = 16$.

LET $\varepsilon > 0$ BE GIVEN.

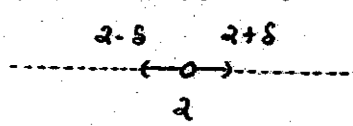
NOTE : WE MUST FIND A $\delta > 0$ S.T.

$0 < |x-2| < \delta \Rightarrow |x^4 - 16| < \varepsilon$.

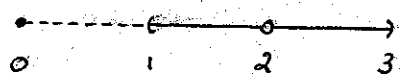
NOTE THAT

$$\begin{aligned} |x^4-16| &= |(x^2+4)(x^2-4)| = |(x^2+4)(x+2)(x-2)| \\ &= |x^2+4||x+2||x-2| \\ &\leq (|x|^2+2)(|x|+2)|x-2| \end{aligned}$$

<u>STRATEGY</u> : WE ARE LOOKING FOR AN INTERVAL $U_\delta'(2)$



ON WHICH $|x^4-16| < \varepsilon$. IT CAN BE AS SMALL AS WE LIKE SO THERE IS NO HARM IN DECIDING AT THE OUTSET THAT WE WILL ONLY CONSIDER x THAT ARE WITHIN 1 of $x_0 = 2$.



ANY SUCH x SATISFIES $|x| < 3$ SO

$$|x|^2 + 4 < 3^2 + 4 = 13$$

AND

$$|x| + 2 < 3 + 2 = 5$$

SO

$$(|x|^2+4)(|x|+2) < (13)(5) = 65$$

FOR THESE x WE THEREFORE HAVE

$$\begin{aligned} |x^4 - 16| &\le (|x|^2+4)(|x+2|)|x-2| \\ &< 65\,|x-2| \end{aligned}$$

AND THIS WILL BE $< \varepsilon$ IF $|x-2| < \frac{\varepsilon}{65}$.

THUS, WE LET $\delta = \min\left(1, \frac{\varepsilon}{65}\right)$. THEN $0 < |x-2| < \delta \Rightarrow$

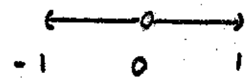
$$\begin{aligned} |x^4-16| &< 65\,|x-2| && \text{BECAUSE } \delta < 1 \\ &< 65\left(\frac{\varepsilon}{65}\right) = \varepsilon && \text{BECAUSE } \delta < \frac{\varepsilon}{65}. \end{aligned}$$

SOMETIMES THE SORT OF ESTIMATES WE HAVE JUST DONE NEED TO "GO THE OTHER WAY", AS WE WILL NOW SEE.

3. $f(x) = \frac{1}{x^2-1}$, $x_0 = 0$

DOMAIN: $\mathbb{R} - \{-1, 1\}$

f IS DEFINED ON THE DELETED NBD $U_1'(0)$ OF 0.



WE SHOW THAT $\lim_{x \to 0} f(x) = -1$.

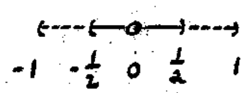
LET $\varepsilon > 0$ BE GIVEN. FOR $x \in U_1'(0)$,

$$|f(x) - (-1)| = \left| \frac{1}{x^2-1} + 1 \right| = \left| \frac{1 + (x^2-1)}{x^2-1} \right| = \frac{|x|^2}{|x^2-1|}$$

$$= \frac{1}{|x-1||x+1|} |x|^2$$

AS IN THE PREVIOUS EXAMPLE WE WOULD LIKE TO ESTIMATE THE SIZE OF $\frac{1}{|x-1||x+1|}$ TO END UP WITH AN INEQUALITY OF THE FORM $|f(x) - (-1)| \leq M|x|^2$ WHICH WE CAN MAKE SMALL BY CHOOSING $|x|$ SMALL. THIS TIME, HOWEVER, $|x-1|$ AND $|x+1|$ APPEAR IN THE DENOMINATOR SO WE MUST REPLACE EACH WITH SOMETHING SMALLER RATHER THAN LARGER.

NOTICE THAT ON $U_{\frac{1}{2}}'(0)$



WE HAVE

$$|x-1| > \frac{1}{2}$$

AND

$$|x+1| > \frac{1}{2}$$

CONSEQUENTLY,

$$|f(x) - (-1)| = \frac{1}{|x-1|\,|x+1|}\,|x|^2$$

$$< \frac{1}{(\frac{1}{2})(\frac{1}{2})}\,|x|^2 = 4\,|x|^2$$

WHICH WILL BE $< \varepsilon$ IF $|x| < \frac{\sqrt{\varepsilon}}{2}$. THUS, WE TAKE

$$\delta = \min\left(\tfrac{1}{2}, \tfrac{\sqrt{\varepsilon}}{2}\right)$$

AND CONCLUDE THAT

$$0 < |x-0| < \delta \Rightarrow |f(x) - (-1)| < 4|x|^2 < 4\left(\frac{\sqrt{\varepsilon}}{2}\right)^2 = \varepsilon$$

AS REQUIRED.

4. DEFINE $f : \mathbb{R} \to \mathbb{R}$ BY $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$

(CALLED THE CHARACTERISTIC FUNCTION OF THE RATIONALS)

LET x_0 BE ANY POINT IN $\mathbb{R}$.

WE SHOW THAT $\lim_{x \to x_0} f(x)$ DOES NOT EXIST.

WE WILL SHOW THAT NO REAL NUMBER L CAN SATISFY THE CONDITION THAT

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

THUS, WE LET L BE AN ARBITRARY REAL NUMBER. WE MUST FIND AN $\varepsilon > 0$ FOR WHICH NO SUCH $\delta > 0$ EXISTS.

L MAY BE 0, OR IT MAY BE 1, OR IT MAY BE NEITHER. IN ANY CASE, WE CAN FIND AN $\varepsilon > 0$ FOR WHICH

$$U_\varepsilon(L) = (L - \varepsilon, L + \varepsilon)$$

FAILS TO CONTAIN AT LEAST ONE OF 0 OR 1.

BUT, SINCE BOTH THE RATIONALS AND IRRATIONALS ARE DENSE IN $\mathbb{R}$, ANY $U_\delta'(x_0)$ WILL CONTAIN AN $x_1 \in \mathbb{Q}$ $(f(x_1) = 1)$ AND AN $x_2 \notin \mathbb{Q}$ $(f(x_2) = 0)$ SO $f(U_\delta'(x_0)) \subseteq U_\varepsilon(L)$ IS IMPOSSIBLE.

<u>EXERCISE 1</u>: DEFINE $f : \mathbb{R} \to \mathbb{R}$ BY

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}.$$

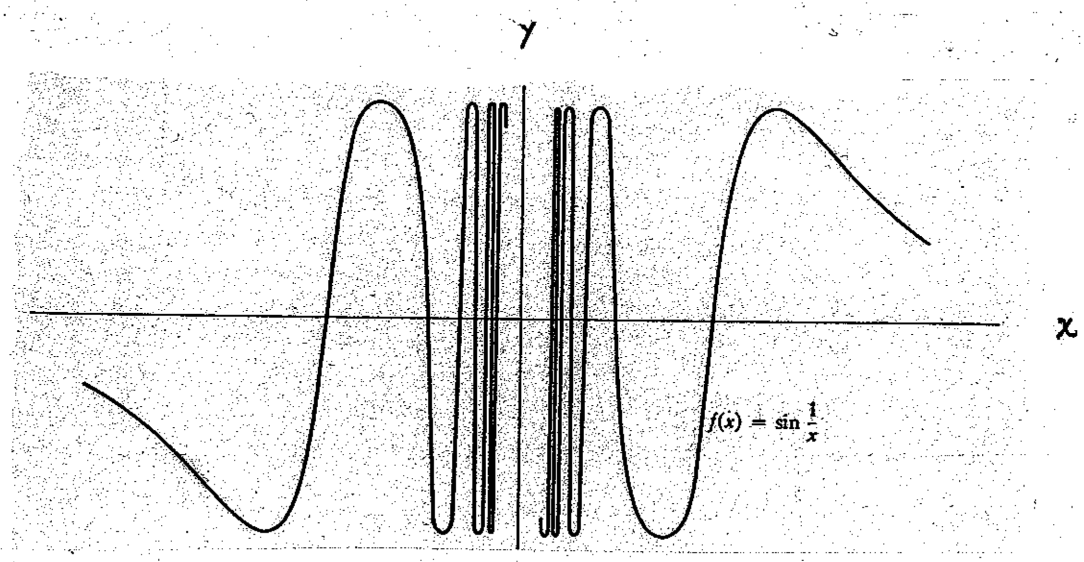
SHOW THAT $\lim_{x \to 0} f(x) = 0$, BUT, IF $x_0 \neq 0$, $\lim_{x \to x_0} f(x)$ DOES NOT EXIST.

<u>EXERCISE 2</u>: DEFINE $f : \mathbb{R} - \{0\} \to \mathbb{R}$ BY $f(x) = \frac{|x|}{x}$. SHOW THAT $\lim_{x \to 0} f(x)$ DOES NOT EXIST.

EXERCISE 3 : DEFINE $f : \mathbb{R} - \{0\} \to \mathbb{R}$ BY $f(x) = \frac{1}{x^2}$. SHOW THAT $\lim_{x \to 0} f(x)$ DOES NOT EXIST.

NOTE : WE WOULD LIKE TO INCLUDE SOME EXAMPLE INVOLVING THE FAMILIAR TRANSCENDENTAL FUNCTIONS $\sin x$, $\cos x$, e^x, $\ln x$, ETC. PLAYING STRICTLY BY THE RULES, WE SHOULD FIRST PRESENT PRECISE DEFINITIONS OF THESE FUNCTIONS. WE WILL EVENTUALLY PROVIDE SUCH DEFINITIONS, BUT, UNFORTUNATELY, THIS REQUIRES TECHNIQUES (FROM CALCULUS) THAT WE HAVE NOT YET DEVELOPED. RATHER THAN BE DENIED SOME VERY INTERESTING AND USEFUL EXAMPLES WE INTEND TO TAKE THE LIBERTY OF USING THESE FUNCTIONS AND THEIR FAMILIAR PROPERTIES BEFORE WE HAVE ANY REAL RIGHT TO DO SO.

5. DEFINE $f : \mathbb{R} - \{0\} \to \mathbb{R}$ BY $f(x) = \sin(\frac{1}{x})$. THE GRAPH OF f IS SHOWN BELOW.



WE SHOW THAT $\lim_{x \to 0} f(x)$ DOES NOT EXIST. TO SEE THIS WE LET L BE AN ARBITRARY REAL NUMBER. WE CAN FIND AN $\varepsilon > 0$ SUCH THAT $U_\varepsilon(L)$ FAILS TO CONTAIN AT LEAST ONE OF 1 OR -1. WE SHOW THAT NO $\delta > 0$ CAN SATISFY $f(U_\delta'(0)) \subseteq U_\varepsilon(L)$ BECAUSE EVERY $U_\delta'(0)$ CONTAINS A POINT x_1 WITH $f(x_1) = 1$ AND ALSO A POINT x_2 WITH $f(x_2) = -1$.

THUS, LET $\delta > 0$ BE ARBITRARY. CHOOSE $n \in \mathbb{N}$ SUCH THAT $\frac{1}{n} < \delta$. (ARCHIMEDIAN PRINCIPLE). THEN

$$x_1 = \frac{1}{\frac{\pi}{2} + 2n\pi} < \frac{1}{n} < \delta$$

AND

$$x_2 = \frac{1}{\frac{3\pi}{2} + 2n\pi} < \frac{1}{n} < \delta$$

SO $x_1, x_2 \in U_\delta'(0)$ AND

$$f(x_1) = \sin\left(\frac{\pi}{2} + 2n\pi\right) = 1$$

$$f(x_2) = \sin\left(\frac{3\pi}{2} + 2n\pi\right) = -1$$

AS REQUIRED.

OUR NEXT EXAMPLE IS A BIT MORE SUBTLE, BUT IT WILL PROVIDE IMPORTANT INFORMATION WHEN WE DISCUSS CONTINUITY AND INTEGRABILITY.

6. (THE DIRICHLET FUNCTION)

FOR THIS EXAMPLE WE WILL AGREE TO WRITE ALL RATIONAL NUMBERS IN THE FORM $\frac{m}{n}$, WHERE m AND n HAVE NO COMMON FACTORS > 1 AND $n > 0$.

DEFINE $f : \mathbb{R} \to \mathbb{R}$ BY

$$f(x) = \begin{cases} \frac{1}{n} , & x = \frac{m}{n} \in \mathbb{Q} \\ 0 , & x \notin \mathbb{Q} \end{cases}$$

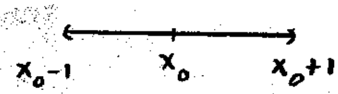
E.G., $f(0) = f\left(\frac{0}{1}\right) = \frac{1}{1} = 1$, $f(1) = f\left(\frac{1}{1}\right) = \frac{1}{1} = 1$, $f\left(-\frac{9}{6}\right) = f\left(\frac{-3}{2}\right) = \frac{1}{2}$, ETC. (NOTE THAT $f(x) \geq 0$ FOR ALL $x \in \mathbb{R}$.)

LET $x_0 \in \mathbb{R}$ BE ARBITRARY. WE CLAIM THAT

$$\lim_{x \to x_0} f(x) = 0.$$

TO PROVE THIS WE FIRST NEED AN OBSERVATION.

FIX $n_0 \in \mathbb{N}$. NOW LET $x_0 \in \mathbb{R}$ BE ARBITRARY AND CONSIDER $U_1(x_0) = (x_0 - 1, x_0 + 1)$.



WE CLAIM THAT $U_1(x_0)$ CONTAINS ONLY FINITELY MANY RATIONAL NUMBERS OF THE FORM $\frac{m}{n_0}$ $(m \in \mathbb{Z})$. INDEED, SUPPOSE THAT

$$x_0 - 1 < \frac{m}{n_0} < x_0 + 1 .$$

THEN

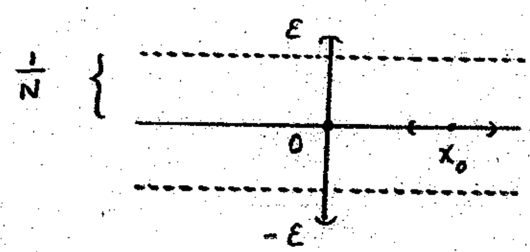
$$n_0(x_0-1) < m < n_0(x_0+1)$$

BUT THERE ARE ONLY FINITELY MANY INTEGERS m IN ANY INTERVAL (a,b), $a,b \in \mathbb{R}$, $a<b$, SO THE CLAIM FOLLOWS.

NOW, TO SHOW THAT $\lim\limits_{x \to x_0} f(x) = 0$ WE LET $\varepsilon > 0$ BE GIVEN.

<u>NOTE</u>: WE MUST FIND A $\delta > 0$ S.T. $0 < |x-x_0| < \delta$ $\Rightarrow |f(x)-0| = f(x) < \varepsilon$. THIS, OF COURSE, IS AUTOMATIC FOR $x \in \mathbb{R}-\mathbb{Q}$ SINCE $f(x)=0$.

CHOOSE SOME $N \in \mathbb{N}$ WITH $\frac{1}{N} < \varepsilon$.



FOR EACH $n_0 = 1, \dots, N-1$, THE INTERVAL (x_0-1, x_0+1) CONTAINS ONLY FINITELY MANY RATIONALS $\frac{m}{n_0}$ WITH DENOMINATOR n_0.

THUS, (x_0-1, x_0+1) CONTAINS ONLY FINITELY MANY $\frac{m}{n}$ WITH $n < N$. LET δ BE THE LEAST POSITIVE DISTANCE FROM x_0 TO ANY OF THESE. THEN $U_\delta(x_0) = (x_0-\delta, x_0+\delta)$ CONTAINS NONE OF THESE EXCEPT PERHAPS x_0 SO $U'_\delta(x_0)$ CONTAINS NONE OF THEM AT ALL. THUS,

$$0 < |x-x_0| < \delta \Rightarrow |f(x)-0| = f(x) \text{ WHICH IS EITHER } 0$$
$$\text{OR } \frac{1}{n} \text{ FOR SOME } n \geq N$$
$$\leq \frac{1}{N} < \varepsilon.$$

THE COMPUTATION OF LIMITS IS GREATLY FACILITATED BY USING CERTAIN GENERAL FACTS CALLED "LIMIT THEOREMS". WE INCORPORATE THESE INTO THE FOLLOWING THEOREM, SOME OF WHICH WE WILL PROVE AND SOME OF WHICH YOU WILL PROVE.

<u>THEOREM</u>: LET f, g AND h BE FUNCTIONS, ALL DEFINED ON SOME DELETED NEIGHBORHOOD OF $x_0 \in \mathbb{R}$. THEN

(1) IF $\lim\limits_{x \to x_0} f(x)$ EXISTS, THEN THE LIMIT IS UNIQUE.

(2) IF $\lim\limits_{x \to x_0} f(x) = L_1$ AND $\lim\limits_{x \to x_0} g(x) = L_2$, THEN

$$\lim_{x \to x_0} (f(x) \pm g(x)) = L_1 \pm L_2$$

AND

$$\lim_{x \to x_0} (f(x) g(x)) = L_1 L_2 .$$

(3) IF $\lim\limits_{x \to x_0} f(x) = L_1$, $g(x)$ IS NONZERO ON SOME DELETED NEIGHBORHOOD OF x_0, $\lim\limits_{x \to x_0} g(x) = L_2$ AND $L_2 \neq 0$, THEN

$$\lim_{x \to x_0} \left(\frac{f(x)}{g(x)} \right) = \frac{L_1}{L_2} .$$

(4) IF $f(x) \leq g(x) \leq h(x)$ FOR ALL x IN SOME DELETED NEIGHBORHOOD OF x_0 AND $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} h(x) = L$, THEN

$$\lim_{x \to x_0} h(x) = L$$

AS WELL,

PROOF OF (1) :

SUPPOSE $\lim_{x \to x_0} f(x) = L_1$ AND $\lim_{x \to x_0} f(x) = L_2$. WE SHOW THAT $L_1 = L_2$.

LET $\varepsilon > 0$ BE ARBITRARY. THEN $\exists\, \delta_1 > 0$ S.T. $0 < |x - x_0| < \delta_1 \Rightarrow$ $|f(x) - L_1| < \frac{\varepsilon}{2}$ AND $\exists\, \delta_2 > 0$ S.T. $0 < |x - x_0| < \delta_2 \Rightarrow$ $|f(x) - L_2| < \frac{\varepsilon}{2}$. LET $\delta = \min(\delta_1, \delta_2)$. THEN $0 < |x - x_0| < \delta \Rightarrow$ $|f(x) - L_1| < \frac{\varepsilon}{2}$ AND $|f(x) - L_2| < \frac{\varepsilon}{2}$ SO

$$\begin{aligned} |L_1 - L_2| &= |f(x) - L_2 + L_1 - f(x)| \\ &\leq |f(x) - L_2| + |L_1 - f(x)| = |f(x) - L_2| + |f(x) - L_1| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

BUT $\varepsilon > 0$ IS ARBITRARY SO THIS IMPLIES THAT $|L_1 - L_2| = 0$ (OTHERWISE, TAKING $\varepsilon = |L_1 - L_2|$ WOULD GIVE $|L_1 - L_2| < |L_1 - L_2|$ WHICH IS IMPOSSIBLE). THUS, $L_1 = L_2$. □

PROOF OF (3) :

LET $\varepsilon > 0$ BE GIVEN. THEN $\exists\, \delta_1 > 0$ S.T. $f(x)$ AND $g(x)$ ARE DEFINED AND $g(x)$ IS NONZERO FOR $0 < |x - x_0| < \delta_1$. FOR ALL SUCH x,

$$\left| \frac{f(x)}{g(x)} - \frac{L_1}{L_2} \right| = \left| \frac{L_2 f(x) - L_1 g(x)}{L_2\, g(x)} \right| = \left| \frac{(L_2 f(x) - L_1 L_2) + (L_1 L_2 - L_1 g(x))}{L_2\, g(x)} \right|$$

$$\leq \frac{|L_2|\,|f(x) - L_1| + |L_1|\,|g(x) - L_2|}{|L_2|\,|g(x)|}$$

THUS,

$$\left| \frac{f(x)}{g(x)} - \frac{L_1}{L_2} \right| \leq \frac{1}{|L_2||g(x)|} \left(|L_2||f(x) - L_1| + |L_1||g(x) - L_2| \right)$$

NOW WE MUST BOUND $\frac{1}{|g(x)|}$ BY A CONSTANT. SINCE $L_2 \neq 0$, $\frac{|L_2|}{2} > 0$ SO $\exists\ \delta_2 > 0$ S.T. $0 < |x - x_0| < \delta_2 \Rightarrow$

$$|g(x) - L_2| < \frac{|L_2|}{2}$$

$$|L_2| - |g(x)| \leq |L_2 - g(x)| < \frac{|L_2|}{2}$$

$$-|g(x)| < \frac{|L_2|}{2} - |L_2| = -\frac{|L_2|}{2}$$

$$|g(x)| > \frac{|L_2|}{2}$$

$$\frac{1}{|g(x)|} < \frac{2}{|L_2|}$$

THUS, IF $\delta_3 = \min(\delta_1, \delta_2)$, $0 < |x - x_0| < \delta_3 \Rightarrow$

$$\left| \frac{f(x)}{g(x)} - \frac{L_1}{L_2} \right| < \frac{2}{|L_2|^2} \left(|L_2||f(x) - L_1| + |L_1||g(x) - L_2| \right)$$

$$\left| \frac{f(x)}{g(x)} - \frac{L_1}{L_2} \right| < \frac{2}{|L_2|}|f(x) - L_1| + \frac{2|L_1|}{|L_2|^2}|g(x) - L_2|$$

NOW, $\exists\ \delta_4 > 0$ S.T. $0 < |x - x_0| < \delta_4 \Rightarrow |f(x) - L_1| < \frac{|L_2|\varepsilon}{4}$ AND $\exists\ \delta_5 > 0$ S.T. $0 < |x - x_0| < \delta_5 \Rightarrow |g(x) - L_2| < \frac{|L_2|^2\varepsilon}{4|L_1|}$.

THUS, IF $\delta = \min(\delta_3, \delta_4, \delta_5)$, THEN $0 < |x - x_0| < \delta \Rightarrow$

$$\left| \frac{f(x)}{g(x)} - \frac{L_1}{L_2} \right| < \frac{2}{|L_2|}\left(\frac{|L_2|\varepsilon}{4} \right) + \frac{2|L_1|}{|L_2|^2}\left(\frac{|L_2|^2\varepsilon}{4|L_1|} \right) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

AS REQUIRED. $\square$

EXERCISE 4 : PROVE (2).

EXERCISE 5 : PROVE, BY INDUCTION, THAT IF
$\lim_{x \to x_0} f_i(x) = L_i$; FOR $i = 1, \dots, n$, THEN

$$\lim_{x \to x_0} (f_1(x) + \cdots + f_n(x)) = L_1 + \cdots + L_n$$

AND

$$\lim_{x \to x_0} (f_1(x) \cdots f_n(x)) = L_1 \cdots L_n .$$

PROOF OF (4) :

THERE IS A $\delta_1 > 0$ S.T. $0 < |x - x_0| < \delta_1 \Rightarrow f(x) \le g(x) \le h(x)$.

NOW LET $\varepsilon > 0$ BE GIVEN. $\exists\ \delta_2 > 0$ AND $\exists\ \delta_3 > 0$ S.T.

$$0 < |x - x_0| < \delta_2 \Rightarrow |f(x) - L| < \varepsilon$$

$$\Rightarrow -\varepsilon < f(x) - L < \varepsilon$$

AND

$$0 < |x - x_0| < \delta_3 \Rightarrow |h(x) - L| < \varepsilon$$

$$\Rightarrow -\varepsilon < h(x) - L < \varepsilon$$

LET $\delta = \min(\delta_1, \delta_2, \delta_3)$. THEN, FOR $0 < |x - x_0| < \delta$,

$$f(x) \le g(x) \le h(x) \Rightarrow f(x) - L \le g(x) - L \le h(x) - L$$

$$\Rightarrow$$

$$-\varepsilon < f(x) - L \le g(x) - L \le h(x) - L < \varepsilon$$

$$-\varepsilon < \qquad g(x) - L \qquad < \varepsilon$$

$$\Rightarrow |g(x) - L| < \varepsilon \quad \text{SO } \lim_{x \to x_0} g(x) = L . \qquad \square$$

EXERCISE 6 : SHOW THAT IF $P(x) = a_n x^n + \cdots + a_1 x + a_0$ IS A POLYNOMIAL AND x_0 IS ANY REAL NUMBER, THEN

$$\lim_{x \to x_0} P(x) = P(x_0).$$

EXERCISE 7 : SHOW THAT IF $R(x) = \frac{P(x)}{Q(x)}$ IS A RATIONAL FUNCTION (I.E., A QUOTIENT OF TWO POLYNOMIALS) AND $x_0 \in \mathbb{R}$ IS SUCH THAT $Q(x_0) \neq 0$, THEN

$$\lim_{x \to x_0} R(x) = R(x_0).$$

EXERCISE 8 : SKETCH A GRAPH OF $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ AND FIND $\lim_{x \to 0} f(x)$. PROVE THAT YOUR ANSWER IS CORRECT.

THERE IS ONLY ONE WAY FOR A LIMIT TO EXIST, BUT WE HAVE SEEN A NUMBER OF WAYS IN WHICH A LIMIT CAN FAIL TO EXIST (COMPARE $\lim_{x \to 0} \frac{1}{x^2}$, $\lim_{x \to 0} \frac{|x|}{x}$ AND $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$). WE WILL NOW INTRODUCE A FEW CONCEPTS THAT HELP DISTINGUISH BETWEEN THESE VARIOUS POSSIBILITIES AND ALSO ENLARGE THE CLASS OF "LIMITING BEHAVIORS" WE CAN STUDY.

THE THEOREM WE JUST PROVED IS EQUALLY TRUE FOR THE NEW TYPES OF LIMITS THAT WE NOW INTRODUCE AND THE PROOFS ARE VIRTUALLY IDENTICAL SO WE WILL GENERALLY NOT WRITE THEM OUT.

$f(x) = \frac{1}{x^2}$ "BLOWS UP" (I.E., BECOMES ARBITRARILY LARGE) AS $x \to 0$ SO $\lim_{x \to 0} \frac{1}{x^2}$ DOES NOT EXIST. EVEN SO, IT IS COMMON TO ABBREVIATE THIS ENTIRE STATE OF AFFAIRS BY WRITING "$\lim_{x \to 0} \frac{1}{x^2} = \infty$". MORE PRECISELY,

LET f BE DEFINED (AT LEAST) ON SOME DELETED NBD OF x_0. THEN THE STATEMENT

$$\lim_{x \to x_0} f(x) = \infty$$

MEANS THAT

$$\forall M > 0 \ \exists \delta > 0 \text{ S.T. } 0 < |x - x_0| < \delta \Rightarrow f(x) > M$$

(SO, IN PARTICULAR, $\lim_{x \to x_0} f(x)$ DOES NOT EXIST).

E.G., LET $f(x) = \frac{1}{x^2}$. LET $M > 0$ BE GIVEN.
THEN $0 < |x - 0| < \frac{1}{\sqrt{M}} \Rightarrow f(x) = \frac{1}{x^2} = \frac{1}{|x|^2} > (\sqrt{M})^2 = M$.

SIMILARLY, THE STATEMENT

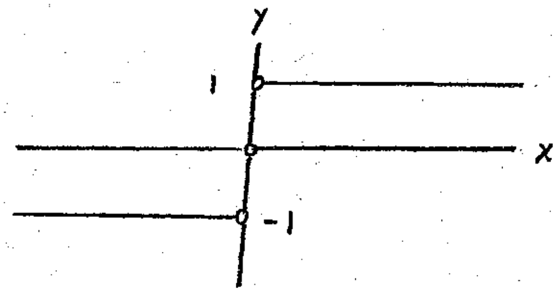
$$\lim_{x \to x_0} f(x) = -\infty$$

MEANS THAT

$$\forall M < 0 \ \exists \delta > 0 \text{ S.T. } 0 < |x - x_0| < \delta \Rightarrow f(x) < M.$$

(SO, IN PARTICULAR, $\lim_{x \to x_0} f(x)$ DOES NOT EXIST).

FOR $f(x) = \frac{|x|}{x}$, $\lim_{x \to 0} f(x)$ FAILS TO EXIST BECAUSE $f(x)$ APPROACHES DIFFERENT THINGS AS $x \to 0$ FROM THE RIGHT AND FROM THE LEFT.



$$f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

HERE ARE THE APPROPRIATE DEFINITIONS.

SUPPOSE f IS DEFINED (AT LEAST) ON SOME INTERVAL (a, x_0). IF L IS A REAL NUMBER WITH THE PROPERTY THAT

$$\forall \varepsilon > 0 \ \exists \ \delta > 0 \text{ S.T. } x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \varepsilon$$

THEN L IS CALLED <u>THE LIMIT OF</u> $f(x)$ <u>AS</u> x <u>APPROACHES</u> x_0 <u>FROM BELOW</u> AND WE WRITE

$$\lim_{x \to x_0^-} f(x) = L.$$

E.G., $\lim_{x \to 0^-} \frac{|x|}{x} = -1$.

SIMILARLY, IF f IS DEFINED ON SOME INTERVAL (x_0, b), THEN

$$\lim_{x \to x_0^+} f(x) = L$$

MEANS THAT

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } x_0 < x < x_0 + \delta \Rightarrow |f(x) - L| < \varepsilon$$

AND THEN L IS CALLED <u>THE LIMIT OF $f(x)$ AS x APPROACHES x_0 FROM ABOVE</u>.

E.G., $\lim\limits_{x \to 0^+} \dfrac{|x|}{x} = 1$.

<u>EXERCISE 9</u>: COMPUTE $\lim\limits_{x \to -2^+} \dfrac{|x+2|}{x^2+x-2}$ AND $\lim\limits_{x \to -2^-} \dfrac{|x+2|}{x^2+x-2}$.

<u>THEOREM</u>: SUPPOSE f IS DEFINED ON SOME DELETED NBD OF $x_0 \in \mathbb{R}$. THEN $\lim\limits_{x \to x_0} f(x)$ EXISTS IF AND ONLY IF BOTH $\lim\limits_{x \to x_0^-} f(x)$ AND $\lim\limits_{x \to x_0^+} f(x)$ EXIST AND $\lim\limits_{x \to x_0^-} f(x) = \lim\limits_{x \to x_0^+} f(x)$.

<u>EXERCISE 10</u>: PROVE THE THEOREM. $\square$

VARIOUS COMBINATIONS OF THESE IDEAS YIELD DEFINITIONS OF ALL OF THE FOLLOWING:

$$\lim_{x \to x_0^-} f(x) = \infty \qquad \lim_{x \to x_0^+} f(x) = \infty$$

$$\lim_{x \to x_0^-} f(x) = -\infty \qquad \lim_{x \to x_0^+} f(x) = -\infty$$

WE WILL WRITE OUT ONE OF THESE CAREFULLY AND LEAVE IT TO YOU TO FORMULATE THE REST.

LET f BE DEFINED ON SOME INTERVAL (x_0, b). THEN THE STATEMENT

$$\lim_{x \to x_0^+} f(x) = -\infty$$

MEANS THAT

$$\forall M < 0 \ \exists \delta > 0 \text{ s.t. } x_0 < x < x_0 + \delta \Rightarrow f(x) < M.$$

<u>EXAMPLE 17</u>: WE PROVE THAT $\lim_{x \to 1^+} \frac{1}{1-x} = -\infty$.

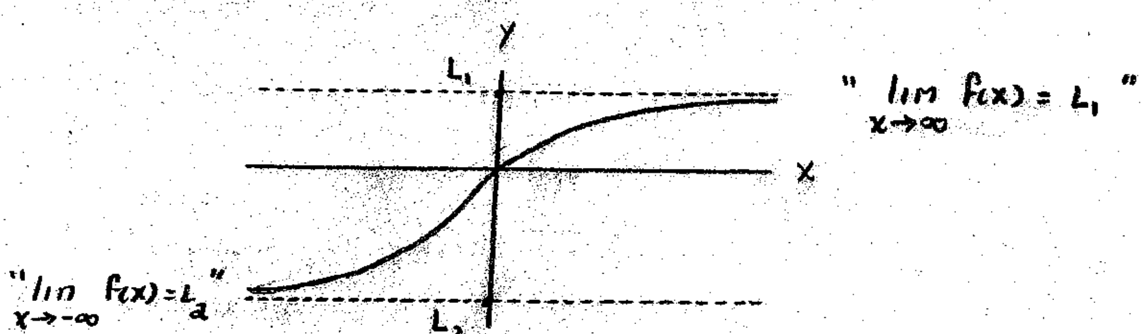
LET $M < 0$ BE GIVEN. THEN $\delta = -\frac{1}{M} > 0$ AND

$1 < x < 1+\delta \Rightarrow 1 < x < 1 - \frac{1}{M} \Rightarrow 1 - x > \frac{1}{M}$

$\Rightarrow \frac{1}{1-x} < M$ AS REQUIRED.

<u>NOTE</u>: FOR A FUNCTION SUCH AS $f(x) = \sqrt{x}$, $\lim_{x \to 0} f(x)$ IS NOT DEFINED SINCE f IS NOT DEFINED ON A DELETED NBD OF 0. HOWEVER, $\lim_{x \to 0^+} \sqrt{x}$ EXISTS (AND, IN FACT, IS 0).

THERE ARE TWO OTHER TYPES OF LIMITING BEHAVIOR THAT WILL BE IMPORTANT TO US. THESE DESCRIBE THE "ASYMPTOTIC" BEHAVIOR OF f, AS ILLUSTRATED BELOW.



HERE ARE THE PRECISE DEFINITIONS.

SUPPOSE f IS DEFINED ON SOME INTERVAL (a,∞). IF L IS A REAL NUMBER WITH THE PROPERTY THAT

$$\forall \varepsilon > 0 \ \exists \ \tau > 0 \text{ s.t. } x > \tau \Rightarrow |f(x) - L| < \varepsilon$$

THEN L IS CALLED <u>THE</u> <u>LIMIT</u> <u>OF</u> $f(x)$ <u>AS</u> x <u>APPROACHES</u> <u>INFINITY</u> AND WE WRITE

$$\lim_{x\to\infty} f(x) = L.$$

SIMILARLY, IF f IS DEFINED ON SOME $(-\infty, a)$, THEN

$$\lim_{x\to-\infty} f(x) = L \iff \forall \varepsilon > 0 \ \exists \ \tau < 0 \text{ s.t. } x < \tau \Rightarrow |f(x) - L| < \varepsilon$$

WE WILL LEAVE IT TO YOU TO FORMULATE THE PRECISE DEFINITIONS OF

$$\lim_{x\to\infty} f(x) = \infty \qquad \lim_{x\to\infty} f(x) = -\infty$$

$$\lim_{x\to-\infty} f(x) = \infty \qquad \lim_{x\to-\infty} f(x) = -\infty$$

<u>EXAMPLES</u>:

8. <u>EXERCISE</u> 11 : SHOW THAT, FOR ANY $n \in \mathbb{N}$, AND ANY CONSTANT k,

$$\lim_{x\to\infty} \frac{k}{x^n} = 0$$

AND

$$\lim_{x\to-\infty} \frac{k}{x^n} = 0.$$

9. THE RESULT OF EXERCISE 11 AND OUR LIMIT THEOREMS MAKE SOME OF THESE COMPUTATIONS QUITE EASY, E.G., IF

$$f(x) = \frac{3x^3-2x^2+7x+8}{2x^3+5}$$

THEN, FOR $x \neq 0$,

$$\frac{3x^3-2x^2+7x+8}{2x^3+5} \; \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \frac{3-\frac{2}{x}+\frac{7}{x^2}+\frac{8}{x^3}}{2+\frac{5}{x^3}}$$

SO

$$\lim_{x\to\pm\infty} \frac{3x^3-2x^2+7x+8}{2x^3+5} = \frac{3}{2}.$$

10. THE TECHNIQUE USED IN EXAMPLE 9 WILL NOT WORK FOR

$$f(x) = \frac{3x^3-2x^2+7x+8}{2x^2+5}$$

(WHY?). WE SHOW DIRECTLY FROM THE DEFINITION THAT

$$\lim_{x\to\infty} \frac{3x^3-2x^2+7x+8}{2x^2+5} = \infty.$$

LET $M > 0$ BE GIVEN. FOR $x > 0$, $7x+8 > 0$ SO

$$\frac{3x^3-2x^2+7x+8}{2x^2+5} > \frac{3x^3-2x^2}{2x^2+5}$$

AND, FOR $x > 1$, $5 < 5x^2$ SO

$$\frac{3x^3-2x^2}{2x^2+5} > \frac{3x^3-2x^2}{2x^2+5x^2} = \frac{3x^3-2x^2}{7x^2} = \frac{3}{7}x - \frac{2}{7}$$

AND THIS WE CAN MAKE LARGER THAN M BY TAKING $x > \frac{7}{3}(M+\frac{2}{7})$. THUS, IF $\tau = \max(1, \frac{7}{3}(M+\frac{2}{7}))$, THEN $x > \tau \Rightarrow f(x) > M$ AS REQUIRED.

ADDITIONAL PROBLEMS :

12. PROVE DIRECTLY FROM THE DEFINITION THAT $\lim_{x\to 1} (x^2+2x+1) = 4$.

13. PROVE DIRECTLY FROM THE DEFINITION THAT $\lim_{x\to 2} \frac{x^3-8}{x-2} = 12$.

14. FIND THE VALUE OF THE FOLLOWING LIMIT AND PROVE DIRECTLY FROM THE DEFINITION THAT YOUR ANSWER IS CORRECT.

$$\lim_{x\to 1} \left(\frac{x^3-1}{x^2-3x+2} + x \right)$$

15. LET $f(x) = c$ FOR EVERY $x \in \mathbb{R}$ BE A CONSTANT FUNCTION. PROVE THAT $\lim_{x\to x_0} f(x) = c$ FOR ANY $x_0 \in \mathbb{R}$.

16. PROVE THAT $\lim_{x\to 0^+} \cos\left(\frac{1}{x}\right)$ AND $\lim_{x\to 0^-} \cos\left(\frac{1}{x}\right)$ DO NOT EXIST.

17. FIND THE VALUES OF $\lim_{x\to 0^+} \frac{x+|x|}{x}$ AND $\lim_{x\to 0^-} \frac{x+|x|}{x}$ AND PROVE THAT YOUR ANSWERS ARE CORRECT.

18. LET $f(x) = x\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right) + \sin\left(\frac{1}{|x|}\right)$. DETERMINE WHETHER OR NOT $\lim_{x\to 0^+} f(x)$ AND $\lim_{x\to 0^-} f(x)$ EXIST AND PROVE THAT YOUR ANSWERS ARE CORRECT.

19. LET $f(x) = \frac{x^2+x-2}{\sqrt{x+2}}$. DETERMINE WHETHER OR NOT $\lim_{x\to -2^-} f(x)$ AND $\lim_{x\to -2^+} f(x)$ EXIST AND PROVE THAT YOUR ANSWERS ARE CORRECT.

20. SUPPOSE $f(x) \geq 0$ FOR x IN (a, x_0) AND $\lim_{x\to x_0^-} f(x)$ EXISTS. PROVE THAT $\lim_{x\to x_0^-} f(x) \geq 0$. IS THE STATEMENT STILL TRUE IF $\geq$ IS REPLACED WITH $>$?

21. PROVE THAT IF $f_1(x) \geq f_2(x)$ FOR x IN (a, x_0) AND $\lim_{x \to x_0^-} f_1(x)$ AND $\lim_{x \to x_0^-} f_2(x)$ BOTH EXIST, THEN $\lim_{x \to x_0^-} f_1(x) \geq \lim_{x \to x_0^-} f_2(x)$.

NOTE: ANALOGUES OF #20 AND #21 FOR $\lim_{x \to x_0^+}$, $\lim_{x \to x_0}$, $\lim_{x \to \infty}$ AND $\lim_{x \to -\infty}$ ARE ALSO TRUE AND ARE PROVED IN THE SAME WAY.

22. SUPPOSE THAT $\lim_{x \to x_0} f(x)$ EXISTS. SHOW THAT f IS BOUNDED ON SOME NBD OF x_0, I.E., THAT THERE EXISTS A CONSTANT M AND A $\rho > 0$ SUCH THAT $0 < |x - x_0| < \rho \Rightarrow |f(x)| \leq M$.

23. SUPPOSE THAT $\lim_{x \to x_0} f(x) = L > 0$. SHOW THAT $f(x) > 0$ ON SOME $U'_\delta(x_0)$ AND THAT $\lim_{x \to x_0} \sqrt{f(x)} = \sqrt{L}$.

24. SHOW THAT $\lim_{x \to \infty} e^{-x} \sin x = 0$ (YOU MAY USE ALL OF THE "USUAL" PROPERTIES OF THESE FUNCTIONS).

25. DETERMINE $\lim_{x \to -\infty} \frac{2|x|}{1+x}$ AND PROVE THAT YOUR ANSWER IS CORRECT.

26. SHOW THAT $\lim_{x \to -\infty} \sin x$ DOES NOT EXIST.

27. SUPPOSE THAT $\lim_{x \to x_0} f(x)$ EXISTS FOR EVERY x_0 IN THE INTERVAL (a, b). SUPPOSE ALSO THAT $g(x) = f(x)$ FOR ALL BUT FINITELY MANY x IN (a, b). SHOW THAT $\lim_{x \to x_0} g(x) = \lim_{x \to x_0} f(x)$ FOR EVERY x_0 IN (a, b).

SOLUTIONS TO THE EXERCISES :

1. $f : \mathbb{R} \to \mathbb{R}$ $\qquad f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$

CLAIM THAT $\lim\limits_{x \to 0} f(x) = 0$:

LET $\varepsilon > 0$ BE GIVEN. TAKE $\delta = \varepsilon$. THEN

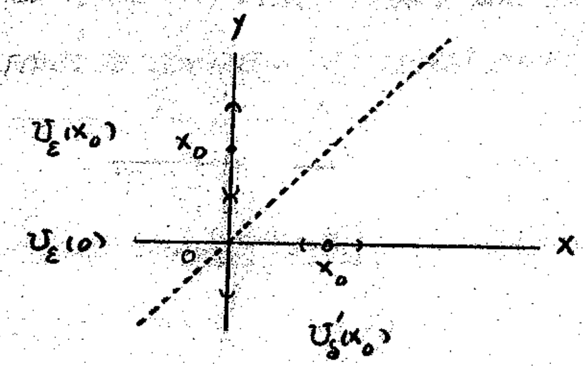
$0 < |x-0| < \delta = \varepsilon \Rightarrow$ $f(x)$ IS EITHER 0 (IF $x \notin \mathbb{Q}$) OR x (IF $x \in \mathbb{Q}$). THUS,

$$|f(x) - 0| = \begin{cases} |x|, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

WHICH IS $< \varepsilon$ IN EITHER CASE.

CLAIM THAT, IF $x_0 \neq 0$, THEN $\lim\limits_{x \to x_0} f(x)$ DOES NOT EXIST.

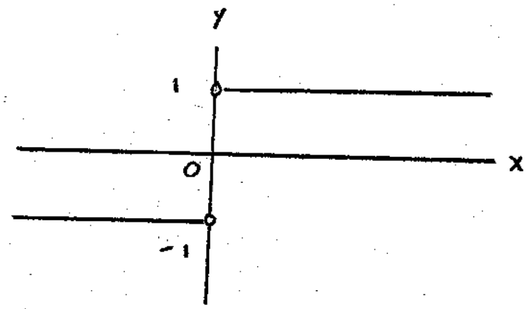
SINCE $x_0 \neq 0$, $\varepsilon = \frac{|x_0|}{2}$ IS POSITIVE



$U_\varepsilon(x_0)$ AND $U_\varepsilon(0)$ ARE DISJOINT BUT EVERY $U'_\delta(x_0)$ CONTAINS POINTS THAT MAP INTO $U_\varepsilon(x_0)$ AND OTHER POINTS THAT MAP INTO $U_\varepsilon(0)$.

IN PARTICULAR, THE LIMIT COULD NOT BE 0 (OR x_0). FOR ANY $L \neq 0$, $U_{|L|}(L)$ DOES NOT CONTAIN 0 AND SO NO $U'_\delta(x_0)$ CAN MAP ENTIRELY INTO IT. THUS, L CANNOT BE THE LIMIT SO THE LIMIT DOES NOT EXIST.

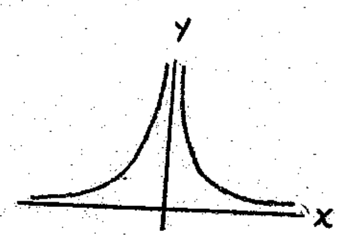
2. $f : \mathbb{R} - \{0\} \to \mathbb{R}$, $f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$



CLAIM THAT $\lim_{x \to 0} f(x)$ DOES NOT EXIST.

FOR ANY REAL NUMBER L WE CAN CHOOSE AN $\varepsilon > 0$ S.T. $U_\varepsilon(L)$ FAILS TO CONTAIN AT LEAST ONE OF -1 OR 1. ANY $U_\delta'(0)$ CONTAINS POINTS WITH $x < 0$ (SO $f(x) = -1$) AND POINTS WITH $x > 0$ (SO $f(x) = 1$) AND THEREFORE $f(U_\delta'(0)) \subseteq U_\varepsilon(L)$ IS IMPOSSIBLE SO L CANNOT BE $\lim_{x \to 0} f(x)$.

3. $f : \mathbb{R} - \{0\} \to \mathbb{R}$, $f(x) = \frac{1}{x^2}$.



CLAIM THAT $\lim_{x \to 0} f(x)$ DOES NOT EXIST.

LET L BE ANY REAL NUMBER, TAKE $\varepsilon = 1$. WE SHOW THAT NO $U_\delta'(0)$ CAN MAP ENTIRELY INTO $U_1(L) = (L-1, L+1)$. SINCE $\mathbb{N}$ IS NOT BOUNDED FROM ABOVE IN $\mathbb{R}$ WE CAN CHOOSE AN $N \in \mathbb{N}$ WITH $N > L+1$. THEN $n \geq N \Rightarrow n^2 \geq N^2 \geq N > L+1$. NOW LET $\delta > 0$ BE ARBITRARY. CHOOSE $m \in \mathbb{N}$ WITH $\frac{1}{m} < \delta$ AND LET $n = \max(m, N)$. THEN $\frac{1}{n} \leq \frac{1}{m} < \delta$ SO $\frac{1}{n} \in U_\delta(0)$, BUT $f(\frac{1}{n}) = n^2 > L+1$ SO $f(\frac{1}{n}) \notin U_1(L)$.

4. ASSUME $\lim_{x \to x_0} f(x) = L_1$ AND $\lim_{x \to x_0} g(x) = L_2$.

CLAIM 1: $\lim_{x \to x_0} (f(x) + g(x)) = L_1 + L_2$

PROOF: LET $\varepsilon > 0$ BE GIVEN. NOTE THAT

$$|(f(x)+g(x)) - (L_1 + L_2)| = |(f(x) - L_1) + (g(x) - L_2)|$$
$$\leq |f(x) - L_1| + |g(x) - L_2|$$

CHOOSE $\delta_1 > 0$ S.T. $0 < |x - x_0| < \delta_1 \Rightarrow |f(x) - L_1| < \frac{\varepsilon}{2}$ AND CHOOSE $\delta_2 > 0$ S.T. $0 < |x - x_0| < \delta_2 \Rightarrow |g(x) - L_2| < \frac{\varepsilon}{2}$. LET $\delta = \min(\delta_1, \delta_2)$. THEN

$$0 < |x - x_0| < \delta \Rightarrow |(f(x) + g(x)) - (L_1 + L_2)| \leq |f(x) - L_1| + |g(x) - L_2|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

AS REQUIRED.

NOTE: THE PROOF THAT $\lim_{x \to x_0} (f(x) - g(x)) = L_1 - L_2$ IS IDENTICAL SINCE

$$|(f(x) - g(x)) - (L_1 - L_2)| = |(f(x) - L_1) + (L_2 - g(x))|$$
$$\leq |f(x) - L_1| + |g(x) - L_2|$$

CLAIM 2: $\lim_{x \to x_0} (f(x)g(x)) = L_1 L_2$

PROOF: LET $\varepsilon > 0$ BE GIVEN. NOTE THAT

$$|f(x)g(x) - L_1 L_2| = |f(x)g(x) - f(x)L_2 + f(x)L_2 - L_1 L_2|$$
$$\leq |f(x)||g(x) - L_2| + |L_2||f(x) - L_1|$$

SINCE $\lim_{x \to x_0} f(x) = L_1$ WE CAN CHOOSE A $\delta_1 > 0$ S.T. $0 < |x - x_0| < \delta_1 \Rightarrow$

$|f(x) - L_1| < 1$ WHICH IMPLIES THAT

$$|f(x)| - |L_1| < 1$$

$$|f(x)| < |L_1| + 1$$

SO

$$|f(x)g(x) - L_1 L_2| < (|L_1|+1)\,|g(x) - L_2| + |L_2|\,|f(x) - L_1|$$

$$< (|L_1|+1)\,|g(x) - L_2| + (|L_2|+1)\,|f(x) - L_1|$$

NOW, $|L_1|+1$ AND $|L_2|+1$ ARE POSITIVE SO WE CAN SELECT A $\delta_2 > 0$ S.T.

$$0 < |x - x_0| < \delta_2 \Rightarrow |g(x) - L_2| < \frac{\varepsilon/2}{|L_1|+1}$$

AND A $\delta_3 > 0$ S.T.

$$0 < |x - x_0| < \delta_3 \Rightarrow |f(x) - L_1| < \frac{\varepsilon/2}{|L_2|+1}.$$

LET $\delta = \min(\delta_1, \delta_2, \delta_3)$. THEN

$$0 < |x - x_0| < \delta \Rightarrow |f(x)g(x) - L_1 L_2| < (|L_1|+1)\left(\frac{\varepsilon/2}{|L_1|+1}\right) + (|L_2|+1)\left(\frac{\varepsilon/2}{|L_2|+1}\right) = \varepsilon$$

AS REQUIRED.

5. WE SHOW THAT, FOR ANY $n \in \mathbb{N}$, $\lim_{x \to x_0} f_i(x) = L_i$ FOR $i = 1, \ldots, n$ IMPLIES

$$\lim_{x \to x_0} (f_1(x) + \cdots + f_n(x)) = L_1 + \cdots + L_n$$

FOR $n = 1$ THERE IS NOTHING TO PROVE AND FOR $n = 2$ THIS WAS PROVED IN EXERCISE 4. NOW LET $k \geq 2$ AND ASSUME THE RESULT FOR A SUM OF k TERMS. THEN IF $\lim_{x \to x_0} f_i(x) = L_i$ FOR $i = 1, \ldots, k, k+1$,

$$\lim_{x\to x_0} (f_1(x)+\cdots+f_k(x)+f_{k+1}(x))$$

$$= \lim_{x\to x_0}(f_1(x)+\cdots+f_k(x)) + \lim_{x\to x_0} f_{k+1}(x) \qquad (n=2 \text{ CASE})$$

$$= \lim_{x\to x_0} f_1(x)+\cdots+\lim_{x\to x_0} f_k(x) + \lim_{x\to x_0} f_{k+1}(x) \qquad (\text{INDUCTION HYPOTHESIS})$$

$$= L_1 + \cdots + L_k + L_{k+1}$$

AS REQUIRED. THE PROOF FOR PRODUCTS IS IDENTICAL (JUST CHANGE EVERY + TO A $\cdot$).

6. THE LIMIT OF A CONSTANT FUNCTION AT ANY POINT IS ITS CONSTANT VALUE AND $\lim_{x\to x_0} x = x_0$. FROM EXERCISE 5 IT THEREFORE FOLLOWS THAT

$$\lim_{x\to x_0} P(x) = \lim_{x\to x_0}(a_n x^n + \cdots + a_1 x + a_0)$$

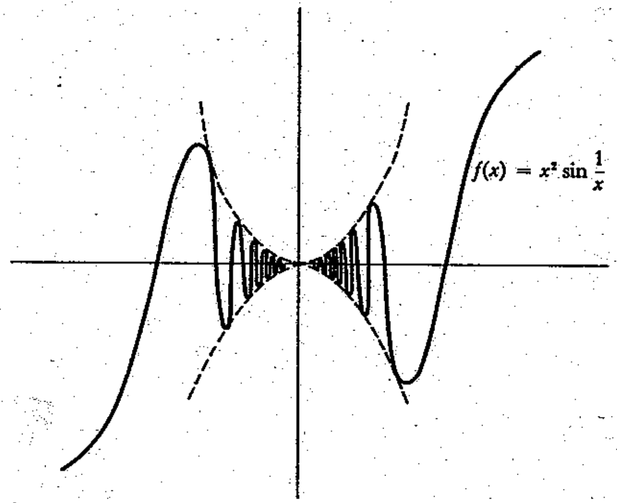
$$= (\lim_{x\to x_0} a_n)(\lim_{x\to x_0} x)^n + \cdots + (\lim_{x\to x_0} a_1)(\lim_{x\to x_0} x) + \lim_{x\to x_0} a_0$$

$$= a_n x_0^n + \cdots + a_1 x_0 + a_0$$

$$= P(x_0)$$

7. THIS FOLLOWS IMMEDIATELY FROM EXERCISE 6 AND PART (3) OF THE THEOREM ON PAGE 15.

8. THE GRAPH OF $f(x) = x^2 \sin(\frac{1}{x})$ IS SHOWN BELOW.



THIS SUGGESTS THAT $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$, WHICH WE PROVE AS FOLLOWS.

LET $\varepsilon > 0$ BE GIVEN. NOTE THAT

$$\left| x^2 \sin\left(\tfrac{1}{x}\right) - 0 \right| = |x|^2 \left|\sin\left(\tfrac{1}{x}\right)\right| \le |x|^2$$

TAKE $\delta = \sqrt{\varepsilon}$. THEN

$$0 < |x - 0| < \delta = \sqrt{\varepsilon} \implies \left| x^2 \sin\left(\tfrac{1}{x}\right) - 0 \right| \le |x|^2 < (\sqrt{\varepsilon})^2 = \varepsilon$$

9. $\lim_{x \to -2^+} \frac{|x+2|}{x^2+x-2}$: FOR $x > -2$, $x+2 > 0$ SO $|x+2| = x+2$ AND

$$\frac{|x+2|}{x^2+x-2} = \frac{x+2}{(x+2)(x-1)} = \frac{1}{x-1}$$

THUS,

$$\lim_{x \to -2^+} \frac{|x+2|}{x^2+x-2} = \lim_{x \to -2^+} \frac{1}{x-1} = \frac{1}{-2-1} = -\frac{1}{3}$$

$\lim_{x \to -2^-} \frac{|x+2|}{x^2+x-2}$: FOR $x < -2$, $x+2 < 0$ SO $|x+2| = -(x+2)$ AND

$$\frac{|x+2|}{x^2+x-2} = \frac{-(x+2)}{(x+2)(x-1)} = -\frac{1}{x-1}$$

THUS,

$$\lim_{x \to -2^-} \frac{|x+2|}{x^2+x-2} = \lim_{x \to -2^-} \left(-\frac{1}{x-1}\right) = -\frac{1}{-2-1} = \frac{1}{3}$$

10. ASSUME FIRST THAT $\lim_{x\to x_0} f(x)$ EXISTS, SAY, $\lim_{x\to x_0} f(x) = L$.

LET $\varepsilon > 0$ BE GIVEN. THEN $\exists\, \delta > 0$ S.T.

$$x_0 - \delta < x < x_0 + \delta \Rightarrow |f(x) - L| < \varepsilon .$$

IN PARTICULAR,

$$x_0 < x < x_0 + \delta \Rightarrow |f(x) - L| < \varepsilon \qquad (\text{so } \lim_{x\to x_0^+} f(x) = L)$$

AND

$$x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \varepsilon \qquad (\text{so } \lim_{x\to x_0^-} f(x) = L)$$

NEXT ASSUME THAT $\lim_{x\to x_0^-} f(x) = L$ AND $\lim_{x\to x_0^+} f(x) = L$. LET $\varepsilon > 0$ BE GIVEN.

THEN $\exists\, \delta_1 > 0$ S.T.

$$x_0 - \delta_1 < x < x_0 \Rightarrow |f(x) - L| < \varepsilon$$

AND $\exists\, \delta_2 > 0$ S.T.

$$x_0 < x < x_0 + \delta_2 \Rightarrow |f(x) - L| < \varepsilon .$$

LET $\delta = \min(\delta_1, \delta_2)$. THEN

$$x_0 - \delta < x < x_0 + \delta \Rightarrow |f(x) - L| < \varepsilon$$

SO $\lim_{x\to x_0} f(x) = L$.

11. $n \in \mathbb{N}$ AND k A CONSTANT. WE SHOW THAT $\lim_{x\to\infty} \frac{k}{x^n} = 0$.

IF $k = 0$ THE FUNCTION IS IDENTICALLY ZERO SO THIS IS CLEAR.

THUS, WE ASSUME $k \neq 0$ AND LET $\varepsilon > 0$ BE GIVEN. NOTE THAT

$$\left| \frac{k}{x^n} - 0 \right| = \frac{|k|}{|x|^n}$$

LET $\tau = \sqrt[n]{\frac{|k|}{\varepsilon}}$. THEN $\tau > 0$ AND

$$x > \tau \Rightarrow |x|^n = x^n > \frac{|k|}{\varepsilon}$$

$$\Rightarrow \frac{1}{|x|^n} < \frac{\varepsilon}{|k|}$$

$$\Rightarrow \left| \frac{k}{x^n} - 0 \right| = \frac{|k|}{|x|^n} < |k| \left(\frac{\varepsilon}{|k|} \right) = \varepsilon$$

NEXT WE SHOW THAT $\lim\limits_{x \to -\infty} \frac{k}{x^n} = 0$. AGAIN, WE ASSUME $k \neq 0$.

LET $\varepsilon > 0$ BE GIVEN. AS ABOVE,

$$\left| \frac{k}{x^n} - 0 \right| = \frac{|k|}{|x|^n}$$

LET $\tau = -\sqrt[n]{\frac{|k|}{\varepsilon}}$. THEN $\tau < 0$ AND

$$x < \tau \Rightarrow |x| > |\tau| = \sqrt[n]{\frac{|k|}{\varepsilon}}$$

$$\Rightarrow |x|^n > \frac{|k|}{\varepsilon}$$

$$\Rightarrow \frac{1}{|x|^n} < \frac{\varepsilon}{|k|}$$

$$\Rightarrow \left| \frac{k}{x^n} - 0 \right| = \frac{|k|}{|x|^n} < |k| \left(\frac{\varepsilon}{|k|} \right) = \varepsilon.$$