ADDENDUM 10

MATHAI - QUILLEN THOM FORM FOR TS^2

HERE WE USE THE UNIVERSAL THOM FORM

$$\nu = (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} (x' + du_1 du_2) \in \Omega^2_{SO(2)}(\mathbb{R}^2)$$

FOR $\mathbb{R}^2$ (SEE APPENDIX 2 AND APPENDIX 4) TO CONSTRUCT REPRESENTATIVES OF THE THOM CLASS FOR TS^2 AND THE EULER CLASS FOR S^2.

FOR THIS WE NEED TO REGARD TS^2 AS THE VECTOR BUNDLE ASSOCIATED TO SOME PRINCIPAL BUNDLE BY A REPRESENTATION OF ITS STRUCTURE GROUP.

PRINCIPAL BUNDLE : $SO(2) \hookrightarrow F_{SO}(TS^2) \xrightarrow{\pi_{SO}} S^2$

REPRESENTATION : $\rho = id_{SO(2)} : SO(2) \to SO(2)$

$$(\text{SO } \rho_* = id_{so(2)})$$

FOR ANY CONNECTION $\omega = \omega' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ON $F_{SO}(TS^2)$ (SEE APPENDIX 1),

$$\Omega = d\omega = d\omega' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \Omega' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

SO

$$x'(\rho_*(\Omega)) = x'(\Omega) = \Omega'$$

(WE WILL MAKE A SPECIFIC CHOICE SHORTLY) SO

KEEP IN MIND THAT ALL FORMS ARE PULLED BACK TO $F_{SO}(TS^2) \times \mathbb{R}^2$ BY PROJECTIONS

$$(2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} (\Omega' + du_1 du_2)$$

INSERT: (FOR THE NEXT THREE PAGES I DROPPED THE NOTATIONAL CONVENTION OF SUPPRESSING THESE PROJECTIONS)

IS A FORM ON $P \times V = F_{SO}(TS^2) \times \mathbb{R}^2$ WHOSE HORIZONTAL PROJECTION (FOR THE CONNECTION $(PR_{F_{SO}(TS^2)})^* \omega$ ON $F_{SO}(TS^2) \times \mathbb{R}^2) \longrightarrow TS^2$) IS BASIC AND DESCENDS TO A REPRESENTATIVE OF THE THOM CLASS FOR TS^2.

WE COMPUTE THIS HORIZONTAL PROJECTION.

NOTE: FOR ANY PRINCIPAL BUNDLE $G \hookrightarrow P \xrightarrow{\pi} X$ WITH CONNECTION ω THERE IS AN EXPLICIT FORMULA FOR COMPUTING THE HORIZONTAL PROJECTION OF A FORM α (WHICH IS DEFINED TO BE THE FORM THAT EVALUATES α ON HORIZONTAL PARTS OF ITS ARGUMENTS). SINCE $\mathfrak{so}(2)$ HAS A SINGLE GENERATOR ξ_1 THIS IS PARTICULARLY SIMPLE: $HOR_\omega(\alpha) = \alpha - \omega^1 \wedge \iota_1 \alpha$, WHERE $\omega = \omega^1 \xi_1$ AND $\iota_1 = \iota_{\xi_1^\#}$ (SEE APPENDIX 3). IN OUR CASE, $P = F_{SO}(TS^2) \times \mathbb{R}^2$ AND ω IS $(PR_{F_{SO}(TS^2)})^* \omega$ WHERE ω IS A CONNECTION ON $SO(2) \hookrightarrow F_{SO}(TS^2) \xrightarrow{\pi_{SO}} S^2$.

LET

$$\alpha = (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2 + u_2^2)} (\Omega^1 + du_1 du_2).$$

WE COMPUTE

$$\begin{aligned} U &= HOR_{(PR_{F_{SO}(TS^2)})^* \omega}(\alpha) \\ &= \alpha - (PR_{F_{SO}(TS^2)})^* \omega^1 \wedge \iota_{\xi_1^\#} \alpha \end{aligned}$$

WHERE $\xi_1^\#$ IS DEFINED FROM THE $SO(2)$-ACTION ON $F_{SO}(TS^2) \times \mathbb{R}^2$ GIVEN BY

$$(p, v) \cdot g = (p \cdot g, g^{-1} \cdot v)$$

WE COMPUTE $\iota_{\xi_1^\#}\alpha$. IDENTIFYING THE TANGENT BUNDLE OF THE PRODUCT MANIFOLD $F_{SO}(TS^2) \times \mathbb{R}^2$ WITH THE SUM OF THE TANGENT BUNDLES OF $F_{SO}(TS^2)$ AND $\mathbb{R}^2$ WE HAVE

$$\xi_1^\#(p,v) = \frac{d}{dt}\left((p,v)\cdot \exp(t\xi_1)\right)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(p\cdot \exp(t\xi_1),\ \exp(-t\xi_1)\cdot v\right)\Big|_{t=0}$$

$$= \xi_{11}^\#(p) + \xi_{12}^\#(v),$$

WHERE WE USE THE EXTRA SUBSCRIPT TO DISTINGUISH THE VECTOR FIELDS ARISING FROM THE RIGHT ACTION OF $SO(2)$ ON $F_{SO}(TS^2)$ AND THE LEFT ACTION OF $SO(2)$ ON $\mathbb{R}^2$. SIMILARLY, FOR THE INTERIOR MULTIPLICATION $\iota_1 = \iota_{\xi_1^\#}$ WE WILL WRITE

$$\iota_1 = \iota_{11} + \iota_{12}$$

SINCE INSERTING $\xi_1^\#$ IN THE 1^{ST} SLOT IS THE SAME AS INSERTING $\xi_{11}^\#$ AND $\xi_{12}^\#$ SEPARATELY AND ADDING.

$$\iota_{\xi_1^\#}\alpha = \iota_{\xi_1^\#}\left((2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}(\Omega' + du_1 du_2)\right)$$

$$= (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}\ \iota_{\xi_1^\#}(\Omega' + du_1 du_2)$$

BECAUSE THE FORMS HAVE BEEN PULLED BACK BY PROJECTIONS.

$$= (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}\left(\underbrace{\iota_{11}\Omega'}_{0 \text{ SINCE } \Omega' \text{ IS HORIZONTAL}} + \iota_{12}(du_1 du_2)\right)$$

$$= (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}(-u_1 du_2 - u_2 du_2)$$

(SEE ADDENDUM 9)

$$\iota_{\xi_1^\#}\alpha = -(2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} (u_1\, du_1 + u_2\, du_2)$$

THUS,

$$U = (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} (\Omega' + du_1 du_2) - (PR_{F_{SO}(TS^2)})^* \omega' \wedge$$
$$(-(2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} (u_1\, du_1 + u_2\, du_2))$$

$$\boxed{U = (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} (\Omega' + du_1 du_2 + (PR_{F_{SO}(TS^2)})^* \omega' \wedge (u_1\, du_1 + u_2\, du_2))}$$

<u>NOTE</u>: $(PR_{F_{SO}(TS^2)})^* \omega'$ OPERATES ON A TANGENT VECTOR V AT (p, v), WRITTEN AS $V = V_1 + V_2$ WITH V_1 TANGENT TO $F_{SO}(TS^2)$ AND V_2 TANGENT TO $\mathbb{R}^2$, TO GIVE $\omega'(V_1)$.

THUS,

$$HOR_{(PR_{F_{SO}(S^2)})^* \omega}(p, v) = HOR_\omega(p) \oplus T_v(\mathbb{R}^2)$$