ADDENDUM 16 : FINITE ENERGY SEIBERG-WITTEN SOLUTIONS

(SW1) $\quad \not{\partial}_A \Psi = 0$

(SW2) $\quad F_A^+ = \sigma^+((\Psi \otimes \Psi^*)_0)$

WE WILL WRITE THEM OUT EXPLICITLY ON $\mathbb{R}^4$ AND PROVE A THEOREM OF WITTEN WHICH ASSERTS THAT ANY SOLUTION (A, Ψ) WITH $\Psi \in L^2(\mathbb{R}^4)$ MUST ACTUALLY HAVE $\Psi \equiv 0$.

THUS, WE LET $M = \mathbb{R}^4$ WITH ITS USUAL RIEMANNIAN METRIC AND ORIENTATION. SINCE $\mathbb{R}^4$ IS CONTRACTIBLE ALL OF THE RELEVANT BUNDLES ARE TRIVIAL AND WE WILL WORK WITH EXPLICIT TRIVIALIZATIONS. THUS, THE ORIENTED, ORTHONORMAL FRAME BUNDLE IS

$$SO(4) \hookrightarrow \mathbb{R}^4 \times SO(4) \longrightarrow \mathbb{R}^4$$

AND THERE IS AN ESSENTIALLY UNIQUE SPIN^c STRUCTURE $\mathcal{L}$

$$\begin{tikzcd} \mathbb{R}^4 \times SPIN^c(4) \arrow[rr, "\Lambda"] \arrow[dr] & & \mathbb{R}^4 \times SO(4) \arrow[dl] \\ & \mathbb{R}^4 & \end{tikzcd}$$

WHERE $\Lambda(x, \xi) = (x, \pi(\xi))$. THE SPINOR BUNDLES ARE THEREFORE ALSO TRIVIAL SO THEIR SECTIONS (I.E., THE SPINOR FIELDS) CAN BE IDENTIFIED WITH GLOBALLY DEFINED FUNCTIONS ON $\mathbb{R}^4$:

$$\Psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix} : \mathbb{R}^4 \longrightarrow S_{\mathbb{C}} = \mathbb{C}^4$$

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ 0 \\ 0 \end{pmatrix} : \mathbb{R}^4 \longrightarrow S^+_{\mathbb{C}} \cong \mathbb{C}^2$$

$$\phi = \begin{pmatrix} 0 \\ 0 \\ \psi^3 \\ \psi^4 \end{pmatrix} : \mathbb{R}^4 \longrightarrow S^-_{\mathbb{C}} \cong \mathbb{C}^2$$

FOR CONVENIENCE WE WILL OFTEN ABUSE THE NOTATION AND WRITE

(1) $$\Psi = \begin{pmatrix} \psi \\ \phi \end{pmatrix}$$

WE WILL USE x^1, x^2, x^3, x^4 FOR THE STANDARD COORDINATES ON $\mathbb{R}^4$ AND WRITE

$$\partial_i = \frac{\partial}{\partial x^i}, \quad i = 1, 2, 3, 4$$

(THESE BEING APPLIED COMPONENTWISE TO SPINOR FIELDS).

THE DETERMINANT LINE BUNDLE OF $\mathcal{L}$ IS LIKEWISE TRIVIAL AS IS THE CORRESPONDING PRINCIPAL $U(1)$-BUNDLE $U(1) \hookrightarrow L^0 \longrightarrow B$:

$$U(1) \hookrightarrow \mathbb{R}^4 \times U(1) \longrightarrow \mathbb{R}^4$$

A $U(1)$-CONNECTION A ON THIS BUNDLE IS THEN UNIQUELY DETERMINED BY A GLOBALLY DEFINED $\mathfrak{u}(1) = \mathrm{Im}\,\mathbb{C}$-VALUED 1-FORM ON $\mathbb{R}^4$:

$$A = A_i \, dx^i$$

$$A_i : \mathbb{R}^4 \rightarrow \mathrm{Im}\,\mathbb{C}, \quad i = 1, 2, 3, 4.$$

NOW, IN ORTHONORMAL COORDINATES THE COVARIANT DERIVATIVE INDUCED BY THE LEVI-CIVITA CONNECTION IS JUST ORDINARY (COMPONENTWISE) EXTERIOR DIFFERENTIATION (CHRISTOFFEL SYMBOLS ARE ZERO) SO THE COVARIANT DERIVATIVE $\nabla = \nabla_A$ INDUCED BY IT AND THE $U(1)$-CONNECTION A TAKES THE FORM $\nabla = d + A$, I.E.,

$$\nabla = \nabla_i \, dx^i = (\partial_i + A_i) dx^i ,$$

SO THAT

$$\nabla \Psi = (\partial_i \Psi + A_i \Psi) dx^i = \begin{pmatrix} (\partial_i \psi^1 + A_i \psi^1) dx^i \\ (\partial_i \psi^2 + A_i \psi^2) dx^i \\ (\partial_i \psi^3 + A_i \psi^3) dx^i \\ (\partial_i \psi^4 + A_i \psi^4) dx^i \end{pmatrix}$$

WITH $\{e_i\} = \{\partial_i\}$ THE STANDARD ORIENTED, ORTHONORMAL FRAME FIELD ON $\mathbb{R}^4$ WE THEREFORE HAVE $\nabla(\Psi)(e_i) = \partial_i \Psi + A_i \Psi$ AND FOR CONVENIENCE WE WILL WRITE THIS

$$\nabla_i \Psi = (\partial_i + A_i) \Psi = \begin{pmatrix} \partial_i \psi^1 + A_i \psi^1 \\ \partial_i \psi^2 + A_i \psi^2 \\ \partial_i \psi^3 + A_i \psi^2 \\ \partial_i \psi^4 + A_i \psi^4 \end{pmatrix}$$

THE DIRAC OPERATOR $\tilde{\not{D}}_A \Psi = \sum_{i=1}^{4} e_i \cdot \nabla_i \Psi$ REQUIRES THAT WE CLIFFORD MULTIPLY BY THE BASIS ELEMENTS e_i, I.E., MATRIX MULTIPLY BY $E_i = T(e_i) \in Cl(4) \otimes_{\mathbb{R}} \mathbb{C}$. FOR THIS WE WILL WRITE $\Psi = \binom{\psi}{\phi}$ AS IN (1) SO THAT

$$\tilde{\not{D}}_A \Psi = \sum_{i=1}^{4} e_i \cdot \nabla_i \Psi = \sum_{i=1}^{4} T(e_i) \nabla_i \Psi$$

$$= \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} \nabla_1 \psi \\ \nabla_1 \phi \end{pmatrix} + \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} \nabla_2 \psi \\ \nabla_2 \phi \end{pmatrix} +$$

$$\begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}\begin{pmatrix} \nabla_3 \psi \\ \nabla_3 \phi \end{pmatrix} + \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}\begin{pmatrix} \nabla_4 \psi \\ \nabla_4 \phi \end{pmatrix}$$

$$= \begin{pmatrix} \nabla_1 \phi + I \nabla_2 \phi + J \nabla_3 \phi + K \nabla_4 \phi \\ -\nabla_1 \psi + I \nabla_2 \psi + J \nabla_3 \psi + K \nabla_4 \psi \end{pmatrix}$$

NOTE THAT, AS EXPECTED, $\not{D}_A$ SENDS POSITIVE SPINORS $\Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$ TO NEGATIVE SPINORS AND NEGATIVE SPINORS $\Psi = \begin{pmatrix} 0 \\ \phi \end{pmatrix}$ TO POSITIVE SPINORS.

SINCE (SW1) INVOLVES ONLY THE RESTRICTION OF $\tilde{\not{D}}_A$ TO POSITIVE SPINOR FIELDS IT WILL BE CONVENIET TO DROP THE EXTRANEOUS ZERO AND WRITE

(2) $$\not{D}_A \psi = -\nabla_1 \psi + I \nabla_2 \psi + J \nabla_3 \psi + K \nabla_4 \psi .$$

(BUT KEEP IN MIND THAT THIS IS A NEGATIVE SPINOR FIELD).

<u>REMARK</u>: IN THE MATHEMATICS (AS OPPOSED TO THE PHYSICS) LITERATURE IT IS COMMON TO REFER TO THIS RESTRICTED OPERATOR AS THE "DIRAC OPERATOR".

TO WRITE OUT (SW1) ON $\mathbb{R}^4$ WE NEED ONLY SET THE EXPRESSION IN (2) EQUAL TO ZERO. FOR (SW2) WE WILL USE EXPRESSION (46), ADDENDUM 14, FOR

$\sigma^+((\psi \otimes \psi^*)_0)$ AND THE FOLLOWING LOCAL DESCRIPTION OF F_A^+ :

WRITE $A = A_i dx^i$. THEN $F_A = dA = \sum_{i<j} F_{ij}\, dx^i \wedge dx^j$, WHERE $F_{ij} = \partial_i A_j - \partial_j A_i$, $i, j = 1, 2, 3, 4$. A BASIS FOR THE SELF-DUAL 2-FORMS IS

$$\{dx^1 \wedge dx^2 + dx^3 \wedge dx^4,\ dx^1 \wedge dx^3 + dx^4 \wedge dx^2,\ dx^1 \wedge dx^4 + dx^2 \wedge dx^3\}$$

AND $F_A^+ = \frac{1}{2}(F_A + {}^*F_A)$ IS GIVEN BY

$$F_A^+ = \tfrac{1}{2}(F_{12} + F_{34})(dx^1 \wedge dx^2 + dx^3 \wedge dx^4) + \tfrac{1}{2}(F_{13} + F_{42})(dx^1 \wedge dx^3 + dx^4 \wedge dx^2) + \tfrac{1}{2}(F_{14} + F_{23})(dx^1 \wedge dx^4 + dx^2 \wedge dx^3)$$

THUS, WE OBTAIN

(SW1) $$\nabla_1 \psi = \nabla_2 \psi + \nabla_3 \psi + \nabla_4 \psi$$

(SW2) $$\begin{aligned} F_{12} + F_{34} &= -\tfrac{1}{2}\, \psi^* I \psi \\ F_{13} + F_{42} &= -\tfrac{1}{2}\, \psi^* J \psi \\ F_{14} + F_{23} &= -\tfrac{1}{2}\, \psi^* K \psi \end{aligned}$$

TO CONVEY SOME SENSE OF WHAT KIND OF EQUATIONS THESE ACTUALLY ARE WE WILL WRITE THEM OUT EXPLICITLY IN TERMS OF THE UNKNOWNS A_1, A_2, A_3, A_4 AND ψ^1, ψ^2 :

$$\begin{pmatrix} -(\partial_1+A_1)+i(\partial_2+A_2) & (\partial_3+A_3)+i(\partial_4+A_4) \\ -(\partial_3+A_3)+i(\partial_4+A_4) & -(\partial_1+A_1)-i(\partial_2+A_2) \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(\partial_1 A_2 - \partial_2 A_1) + (\partial_3 A_4 - \partial_4 A_3) = -\tfrac{1}{2} i \left(|\psi^1|^2 - |\psi^2|^2 \right)$$
$$(\partial_1 A_3 - \partial_3 A_1) + (\partial_4 A_2 - \partial_2 A_4) = -\; i \, \mathrm{Im} (\bar{\psi}^1 \psi^2)$$
$$(\partial_1 A_4 - \partial_4 A_1) + (\partial_2 A_3 - \partial_3 A_2) = -\; i \, \mathrm{Re} (\bar{\psi}^1 \psi^2)$$

NOTICE THAT THE EQUATIONS ARE ONLY RATHER MILDLY NONLINEAR (ON THE RIGHT-HAND SIDE OF (SW2)).

REMARK: IT IS PERHAPS WORTH OBSERVING THAT THESE EQUATIONS DO HAVE NONTRIVIAL SOLUTIONS: LET US TAKE $\psi \equiv 0$ (FOR REASONS WE WILL DISCUSS LATER, A SOLUTION (A, ψ) TO (SW) WITH $\psi \equiv 0$ IS SAID TO BE REDUCIBLE). THEN (SW1) IS, OF COURSE, IDENTICALLY SATISFIED AND (SW2) REDUCES TO THE STATEMENT THAT A IS AN ANTI-SELF-DUAL $U(1)$-CONNECTION. ONE CAN CONSTRUCT SUCH THINGS AS FOLLOWS: LET $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ BE A SMOOTH VECTOR FIELD ON $\mathbb{R}^3$, $\vec{F} = (F_1, F_2, F_3)$. LET $\Delta = -\sum_{i=1}^{3} \frac{\partial^2}{(\partial x^i)^2}$ BE THE USUAL LAPLACIAN ON $\mathbb{R}^3$ AND WRITE $\Delta \vec{F}$ FOR $(\Delta F_1, \Delta F_2, \Delta F_3)$. VECTOR IDENTITIES GIVE

$$\mathrm{CURL}\,(\mathrm{CURL}\,\vec{F}) = -\Delta \vec{F} + \mathrm{GRAD}\,(\mathrm{DIV}\,\vec{F}).$$

NOW CHOOSE $\vec{F}$ TO BE (COMPONENTWISE) HARMONIC WITH NONCONSTANT $\mathrm{GRAD}\,(\mathrm{DIV}\,\vec{F})$ (E.G., $\vec{F}(x^1, x^2, x^3) = (e^{x^1} \cos(x^2), 0, 0)$) AND SET

$$\vec{G} = (G_1, G_2, G_3) = \mathrm{CURL}\,(\mathrm{CURL}\,\vec{F}) = \mathrm{GRAD}\,(\mathrm{DIV}\,\vec{F}).$$

THEN $\vec{G}: \mathbb{R}^3 \to \mathbb{R}^3$ IS NONCONSTANT AND SATISFIES

(3) $$\mathrm{DIV}\,\vec{G} = 0 \quad \text{AND} \quad \mathrm{CURL}\,\vec{G} = \vec{0}$$

NOW THINK OF $\vec{G}$ AS A FUNCTION ON $\mathbb{R}^4$ THAT IS INDEPENDENT OF x^4 AND

DEFINE AN ANTI-SELF-DUAL 2-FORM ω ON $\mathbb{R}^4$ BY

$$\omega = G_1(dx^1\wedge dx^2 - dx^3\wedge dx^4) + G_2(dx^1\wedge dx^3 - dx^4\wedge dx^2) + G_3(dx^1\wedge dx^4 - dx^2\wedge dx^3)$$

IT FOLLOWS FROM (3) THAT $d\omega = 0$ SO, BY THE POINCARÉ LEMMA, THERE IS A 1-FORM η ON $\mathbb{R}^4$ WITH $d\eta = \omega$. THUS, IF WE LET $A = i\eta$ WE HAVE A $U(1)$-CONNECTION FORM WHICH IS ANTI-SELF-DUAL BECAUSE $F_A = dA = i\,d\eta = i\omega$.

OUR OBJECTIVE NOW IS TO PROVE THAT ANY SOLUTION (A, Ψ) TO (SW) ON $\mathbb{R}^4$ FOR WHICH THE SPINOR FIELD Ψ IS SQUARE INTEGRABLE ON $\mathbb{R}^4$ MUST, IN FACT, BE REDUCIBLE.

<u>THEOREM</u> (WITTEN): SUPPOSE A IS IN $\Omega^1(\mathbb{R}^4, \operatorname{Im}\mathbb{C})$ AND $\Psi = \begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}$ IS IN $C^\infty(\mathbb{R}^4, \mathbb{C}^2)$ AND THAT THE PAIR (A, Ψ) SATISFIES

(4) $$\nabla_1\Psi = I\nabla_2\Psi + J\nabla_3\Psi + K\nabla_4\Psi \qquad (\not{\partial}_A\Psi = 0)$$

(5a) $$F_{12} + F_{34} = -\tfrac{1}{2}\Psi^* I\Psi$$

(5b) $$F_{13} + F_{42} = -\tfrac{1}{2}\Psi^* J\Psi$$

(5c) $$F_{14} + F_{23} = -\tfrac{1}{2}\Psi^* K\Psi$$

THEN $\Psi \in L^2(\mathbb{R}^4)$ IMPLIES $\Psi \equiv 0$.

<u>NOTE</u> : $\Psi \in L^2(\mathbb{R}^4)$ MEANS $\int_{\mathbb{R}^4} \|\Psi(x)\|^2\, d\mathrm{vol} < \infty$, WHERE $\|\Psi(x)\|^2$ IS THE SQUARED NORM OF $\Psi(x)$ DETERMINED BY THE USUAL HERMITIAN INNER

PRODUCT ON $\mathbb{C}^2$, I.E., $\|\psi(x)\|^2 = |\psi^1(x)|^2 + |\psi^2(x)|^2$, AND $d\mathrm{vol}$ IS THE USUAL VOLUME FORM ON $\mathbb{R}^4$.

PROOF: LET $\Delta = -\sum_{i=1}^{4} \frac{\partial^2}{(\partial x^i)^2}$ BE THE USUAL LAPLACIAN ON $\mathbb{R}^4$.

THEN

(6) $$\Delta \|\psi\|^2 = -2 \sum_{i=1}^{4} \frac{\partial}{\partial x^i} \mathrm{Re} \langle \psi, \nabla_i \psi \rangle .$$

TO SEE THIS COMPUTE

$$\begin{aligned} \frac{\partial}{\partial x^i} \|\psi\|^2 &= \frac{\partial}{\partial x^i} (\bar{\psi}^1 \psi^1 + \bar{\psi}^2 \psi^2) \\ &= \bar{\psi}_1 \partial_i \psi^1 + \psi_1 \partial_i \bar{\psi}_1 + \bar{\psi}_2 \partial_i \psi^2 + \psi^2 \partial_i \bar{\psi}^2 \end{aligned}$$

AND

$$\begin{aligned} \langle \psi, \nabla_i \psi \rangle &= \langle \psi, \partial_i \psi + A_i \psi \rangle \\ &= \bar{\psi}^1 (\partial_i \psi^1 + A_i \psi^1) + \bar{\psi}^2 (\partial_i \psi^2 + A_i \psi^2) \\ &= \bar{\psi}^1 \partial_i \psi^1 + \bar{\psi}^2 \partial_i \psi^2 + A_i (|\psi^1|^2 + |\psi^2|^2) . \end{aligned}$$

THEN

$$\begin{aligned} 2 \mathrm{Re} \langle \psi, \nabla_i \psi \rangle &= \langle \psi, \nabla_i \psi \rangle + \overline{\langle \psi, \nabla_i \psi \rangle} \\ &= \bar{\psi}^1 \partial_i \psi^1 + \psi^1 \partial_i \bar{\psi}^1 + \bar{\psi}^2 \partial_i \psi^2 + \psi^2 \partial_i \bar{\psi}^2 \end{aligned}$$

BECAUSE $A_i (|\psi^1|^2 + |\psi^2|^2) \in \mathrm{Im}\, \mathbb{C}$. THUS,

$$\frac{\partial}{\partial x^i} \|\psi\|^2 = 2 \mathrm{Re} \langle \psi, \nabla_i \psi \rangle$$

SO

$$-\frac{\partial^2}{(\partial x^i)^2} \|\psi\|^2 = -2 \frac{\partial}{\partial x^i} \mathrm{Re} \langle \psi, \nabla_i \psi \rangle$$

AND SUMMING OVER $i = 1,2,3,4$ GIVES (6). A SIMILAR (BUT SOMEWHAT LONGER) CALCULATION GIVES

(7) $$\frac{\partial}{\partial x^i} \operatorname{Re} \langle \psi, \nabla_i \psi \rangle = \| \nabla_i \psi \|^2 + \operatorname{Re} \langle \psi, \nabla_i \nabla_i \psi \rangle .$$

AT THIS POINT WE REQUIRE A SPECIAL CASE OF THE FAMOUS <u>WEITZENBÖCK FORMULA</u> WHICH, IN OUR PRESENT CIRCUMSTANCES, READS

(8) $$\not{D}_A^* \not{D}_A \psi + \sum_{i=1}^{4} \nabla_i \nabla_i \psi = (\rho^+(F_A))\psi$$

AND CAN BE PROVED BY DIRECT CALCULATION. NOW, USING THE FACT THAT (A, ψ) SATISFIES $\not{D}_A \psi = 0$, THIS BECOMES

$$\sum_{i=1}^{4} \nabla_i \nabla_i \psi = (\rho^+(F_A))\psi$$

AND, WRITING $\rho^+(F_A)$ AS IN (37), ADDENDUM 14, WE OBTAIN

(9) $$\sum_{i=1}^{4} \nabla_i \nabla_i \psi = (F_{12}+F_{34}) I\psi + (F_{13}+F_{42}) J\psi + (F_{14}+F_{23}) K\psi .$$

THUS, SUBSTITUTING (7) INTO (6) AND USING (9) GIVES

(10) $$\Delta \|\psi\|^2 = -2 \sum_{i=1}^{4} \|\nabla_i \psi\|^2 - 2 \operatorname{Re} \langle \psi, (F_{12}+F_{34}) I\psi \rangle$$

$$- 2 \operatorname{Re} \langle \psi, (F_{13}+F_{42}) J\psi \rangle - 2 \operatorname{Re} \langle \psi, (F_{14}+F_{23}) K\psi \rangle$$

NOW WE EXAMINE THE LAST THREE TERMS IN (10). BY (5a),

$$\langle \psi, (F_{12}+F_{34}) I\psi \rangle = \langle \psi, -\tfrac{1}{2}(\psi^* I \psi) I\psi \rangle$$

AND, BY (45) AND (46), ADDENDUM 14, $-\frac{1}{2}(\psi^* I \psi) = -\frac{1}{2} i (|\psi^1|^2 - |\psi^2|^2)$ SO

$$\langle \psi, (F_{12}+F_{34}) I\psi \rangle = \langle \psi, -\tfrac{1}{2} i \,(|\psi^1|^2 - |\psi^2|^2) I\psi \rangle$$

$$= -\tfrac{1}{2} i \,(|\psi^1|^2 - |\psi^2|^2) \langle \psi, I\psi \rangle$$

$$= -\tfrac{1}{2} i \,(|\psi^1|^2 - |\psi^2|^2) \left\langle \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}, \begin{pmatrix} i\psi^1 \\ -i\psi^2 \end{pmatrix} \right\rangle$$

$$= \tfrac{1}{2} \,(|\psi^1|^2 - |\psi^2|^2)^2$$

$$= \tfrac{1}{2} |\psi^* I \psi|^2$$

SIMILARLY, (5b) AND (5c) GIVE

$$\langle \psi, (F_{13}+F_{42}) J\psi \rangle = \tfrac{1}{2} |\psi^* J \psi|^2$$

AND

$$\langle \psi, (F_{14}+F_{23}) K\psi \rangle = \tfrac{1}{2} |\psi^* K \psi|^2 .$$

OUR FINAL EXPRESSION FOR $\Delta \|\psi\|^2$ FOLLOWS FROM THESE AND (10).

$$(11) \qquad \Delta \|\psi\|^2 = -2 \sum_{i=1}^{4} \|\nabla_i \psi\|^2 - |\psi^* I \psi|^2 - |\psi^* J \psi|^2 - |\psi^* K \psi|^2$$

IN PARTICULAR, $\Delta \|\psi\|^2 \leq 0$ ON ALL OF $\mathbb{R}^4$ SO THE FUNCTION

$$x \longrightarrow \|\psi(x)\|^2 \quad : \quad \mathbb{R}^4 \longrightarrow \mathbb{R}$$

IS SUBHARMONIC ON ALL OF $\mathbb{R}^4$.

<u>REMARK</u>: FOR BASIC PROPERTIES OF SUBHARMONIC FUNCTIONS SEE <u>FOUNDATIONS OF MODERN POTENTIAL THEORY</u>, N.S. LANDKOF, SPRINGER-VERLAG, NEW YORK BERLIN 1972

CONSEQUENTLY, THIS FUNCTION SATISFIES A MEAN VALUE PROPERTY ON $\mathbb{R}^4$. SPECIFICALLY, FOR ANY $r > 0$ AND ANY $x \in \mathbb{R}^4$,

$$(12) \qquad \|\psi(x)\|^2 \leq \frac{2}{\pi^2 r^4} \int_{B_r(x)} \|\psi(x)\|^2 \, d\,\mathrm{vol}$$

WHERE $B_r(x)$ IS THE CLOSED BALL OF RADIUS r ABOUT x.

NOW, ASSUMING $\psi \in L^2(\mathbb{R}^4)$, $\int_{\mathbb{R}^4} \|\psi(x)\|^2 d\,\mathrm{vol} < \infty$. DENOTING THE VALUE OF THIS INTEGRAL BY K WE FIND FROM (12) THAT, FOR ANY $x \in \mathbb{R}^4$,

$$\|\psi(x)\|^2 \leq \frac{2K}{\pi^2 r^4}$$

FOR ANY $r > 0$. THUS, $\|\psi(x)\| = 0$ FOR ANY $x \in \mathbb{R}^4$, I.E., $\psi \equiv 0$ AS REQUIRED. $\square$

<u>REMARKS</u> : WE BRIEFLY DESCRIBE A FEW EXTENSIONS AND GENERALIZATIONS FOR COMPACT M. NOTE THAT THE SOLUTIONS (A, ψ) TO THE SEIBERG-WITTEN EQUATIONS OBVIOUSLY GIVE THE ABSOLUTE MINIMA OF THE FUNCTIONAL

$$(13) \qquad \int_M \left(\|\not{D}_A \psi\|^2 + 2 \left| F_A^+ - \sigma^+((\psi \otimes \psi^*)_0) \right|^2 \right) d\,\mathrm{vol}_g$$

(THE 2 IS FOR CONVENIENCE). USING THE GENERAL WEITZENBÖCK FORMULA ONE CAN SHOW THAT THIS FUNCTIONAL CAN BE WRITTEN IN TERMS OF THE <u>SEIBERG-WITTEN ENERGY FUNCTIONAL</u>

(14) $$E(A,\Psi) = \int_M (|\nabla_A \Psi|^2 + \frac{s}{4}|\Psi|^2 + \frac{1}{4}|\Psi|^4 + |F_A^+|^2)\, d\mathrm{vol}_g$$

(s = SCALAR CURVATURE OF (M, g)) AS FOLLOWS :

$$E(A,\Psi) = \int_M (\|\not{\partial}_A \Psi\|^2 + 2\,|F_A^+ - \sigma^+((\Psi\otimes\Psi^*)_0)|^2)\, d\mathrm{vol}_g - \pi^2\langle c_1(L_0)^2, [M]\rangle.$$

SINCE $\pi^2\langle c_1(L^0)^2, [M]\rangle = \int_M (2|F_A^+|^2 - |F_A|^2)\, d\mathrm{vol}_g$ WE FIND THAT, IF (A,Ψ) SATISFIES (SW), THEN

$$\int_M (|\nabla_A \Psi|^2 + \frac{s}{4}|\Psi|^2 + \frac{1}{4}|\Psi|^4 + 2|F_A^+|^2)\, d\mathrm{vol}_g = 0.$$

THUS, IF $s \geq 0$ WE MUST HAVE $\Psi \equiv 0$ AND $F_A^+ \equiv 0$. WE CONCLUDE THAT <u>IF M ADMITS A RIEMANNIAN METRIC WITH NON-NEGATIVE SCALAR CURVATURE, THEN FOR ANY SPIN^c- STRUCTURE ON THE CORRESPONDING ORIENTED, ORTHONORMAL FRAME BUNDLE ANY SOLUTION (A,Ψ) TO (SW) SATISFIES $\Psi \equiv 0$ AND $F_A^+ \equiv 0$.</u>

NOW, THE INTEGRALS IN (13) AND (14) ARE GENERALLY NOT MEANINGFUL (FINITE) ON THE NONCOMPACT MANIFOLD $\mathbb{R}^4$. INDEED, IF WE DEFINE THE ENERGY $E(A,\Psi)$ OF A PAIR (A,Ψ) SATISFYING (4) AND (5a) - (5c) ON $\mathbb{R}^4$ BY (14) (WITH $s = 0$) ONE CAN SHOW THAT FINITE ENERGY IMPLIES NOT ONLY THAT $\Psi \equiv 0$, BUT ALSO THAT A IS FLAT (I.E., $F_A \equiv 0$).

<u>THEOREM</u> (WITTEN) : SUPPOSE A IS IN $\Omega^1(\mathbb{R}^4, \mathrm{Im}\,\mathbb{C})$ AND $\Psi = \begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}$ IS IN $C^\infty(\mathbb{R}^4, \mathbb{C}^2)$ AND THAT THE PAIR (A,Ψ) SATISFIES (SW) ON $\mathbb{R}^4$, I.E., (4) AND (5a) - (5c). LET

(15) $$E(A,\Psi) = \int_{\mathbb{R}^4} (\sum_{i=1}^{4} \|\nabla_i \Psi\|^2 + \frac{1}{4}\|\Psi\|^4 + \sum_{i<j} |F_{ij}|^2)\, d\mathrm{vol}.$$

THEN $E(A,\Psi) < \infty$ IMPLIES $\Psi \equiv 0$ AND $F_A \equiv 0$.

PROOF : WE USE THE SAME NOTATION AS IN THE PROOF OF THE PREVIOUS THEOREM. SINCE THE ASSUMPTION THAT $\psi \in L^2(\mathbb{R}^4)$ WAS NOT USED UNTIL THE LAST PARAGRAPH OF THAT PROOF WE MAY USE THE IDENTITY (11) DERIVED PRIOR TO THIS.

A SIMPLE CALCULATION SHOWS THAT

$$|\psi^* I \psi|^2 + |\psi^* J \psi|^2 + |\psi^* K \psi|^2 = \|\psi\|^4$$

SO (11) BECOMES

(16) $$\Delta \|\psi\|^2 = -2 \sum_{i=1}^{4} \|\nabla_i \psi\|^2 - \|\psi\|^4 .$$

NOW WE APPEAL TO AN IDENTITY FROM VECTOR CALCULUS ($\Delta(fg) = f\Delta g - 2\nabla f \cdot \nabla g + g \Delta f$) TO OBTAIN

$$\begin{aligned} \Delta \|\psi\|^4 &= \Delta(\|\psi\|^2 \|\psi\|^2) \\ &= -2(\nabla \|\psi\|^2) \cdot (\nabla \|\psi\|^2) + 2\|\psi\|^2 \Delta \|\psi\|^2 \end{aligned}$$

WHICH, WITH (16), GIVES

(17) $$\Delta \|\psi\|^4 = -2(\nabla \|\psi\|^2) \cdot (\nabla \|\psi\|^2) - 4\|\psi\|^2 \sum_{i=1}^{4} \|\nabla_i \psi\|^2 - 2\|\psi\|^6 .$$

IN PARTICULAR, $\|\psi\|^4$ IS SUBHARMONIC ON $\mathbb{R}^4$. SINCE $E(A,\psi) < \infty$ IMPLIES $\int_{\mathbb{R}^4} \|\psi\|^4 \, d\mathrm{vol} < \infty$, THE SAME ARGUMENT WE USED FOR $\|\psi\|^2$ IN THE PREVIOUS PROOF SHOWS ONCE AGAIN THAT

(18) $$\psi \equiv 0 .$$

TO SHOW THAT $F_A \equiv 0$ AS WELL WE OBSERVE THAT $\psi \equiv 0$ AND $F_A^+ = \sigma^+((\psi \otimes \psi^*)_0)$ TOGETHER IMPLY THAT $F_A^+ = 0$, I.E., $F = F_A$ IS ANTI-SELF-DUAL. THUS,

(19) $$F = dA = -*dA.$$

WE CLAIM THAT IT FOLLOWS FROM THIS THAT F IS A HARMONIC 2-FORM, I.E.,

(20) $$\Delta_2 F = 0,$$

WHERE Δ_2 IS THE HODGE LAPLACIAN ON 2-FORMS.

<u>REMARK</u>: FOR p-FORMS ON AN ORIENTED, RIEMANNIAN n-MANIFOLD ONE DEFINES $\delta = (-1)^{n(p+1)+1} * d *$ AND THEN THE LAPLACIAN ON FORMS IS DEFINED BY

$$\Delta_p = d \circ \delta + \delta \circ d.$$

FOR 2-FORMS ON $\mathbb{R}^4$, $\delta = -*d*$ SO $\Delta_2 = -(d*d* + *d*d)$. RELATIVE TO STANDARD COORDINATES, $F = \sum_{i<j} F_{ij}\, dx^i \wedge dx^j$ IMPLIES

(21) $$\Delta_2 F = \sum_{i<j} (\Delta F_{ij})\, dx^i \wedge dx^j$$

TO PROVE THAT (19) IMPLIES (20) WE JUST COMPUTE

$$\begin{aligned}
\Delta_2 F &= -(d*d* + *d*d)F = -(d*d* + *d*d)(dA) \\
&= -d*d*(dA) - *d*d(dA) = -d*d*(-*dA) + *d*(d(dA)) \\
&= d*d(**dA) + *d*(d^2A) = d*(d^2A) + *d*(d^2A) \\
&= 0 \quad \text{SINCE } d^2 = 0.
\end{aligned}$$

FROM (21) WE THEN CONCLUDE THAT EACH F_{ij} IS HARMONIC:

$$\Delta F_{ij} = 0 \ , \quad i,j = 1,2,3,4 \ , \quad i<j \ ,$$

NOW, SINCE $F_{ij} : \mathbb{R}^4 \to \operatorname{Im} \mathbb{C}$, $|F_{ij}|^2 = -F_{ij}^2$ SO

$$\begin{aligned}
\Delta \Big(\sum_{i<j} |F_{ij}|^2 \Big) &= - \sum_{i<j} \Delta (F_{ij})^2 \\
&= - \sum_{i<j} \big[-2 \nabla F_{ij} \cdot \nabla F_{ij} + 2 F_{ij} \Delta F_{ij} \big] \\
&= 2 \sum_{i<j} \nabla F_{ij} \cdot \nabla F_{ij} = 2 \sum_{i<j} \Big(\sum_{k=1}^{4} \Big(\frac{\partial F_{ij}}{\partial x^k} \Big)^2 \Big) \\
&= -2 \sum_{i<j} \Big(\sum_{k=1}^{4} \Big(\frac{\partial F_{ij}}{\partial x^k} \Big) \Big(- \frac{\partial F_{ij}}{\partial x^k} \Big) \Big) \\
&= -2 \sum_{i<j} \Big(\sum_{k=1}^{4} \Big| \frac{\partial F_{ij}}{\partial x^k} \Big|^2 \Big) \\
&\leq 0 .
\end{aligned}$$

THUS, $\sum_{i<j} |F_{ij}|^2$ IS SUBHARMONIC ON $\mathbb{R}^4$. BUT $E(A,\psi) < \infty$ IMPLIES

$$\int_{\mathbb{R}^4} \Big(\sum_{i<j} |F_{ij}|^2 \Big) \, d\mathrm{vol} < \infty$$

SO ONCE AGAIN THE MEAN VALUE PROPERTY IMPLIES $\sum_{i<j} |F_{ij}|^2 \equiv 0$.
FROM THIS, EACH $F_{ij} \equiv 0$ SO $F \equiv 0$. $\square$