ADDENDUM 2 : THE MODULI SPACES

M = COMPACT, SIMPLY CONNECTED, ORIENTED, SMOOTH 4-MANIFOLD

+ ASSUMPTIONS ON $b_2^+(M)$ AS WE PROCEED

$$SU(2) \hookrightarrow P_k \xrightarrow{\pi_k} M$$

IS THE PRINCIPAL $SU(2)$-BUNDLE OVER M WITH CHERN CLASS $k > 0$.

$\mathcal{A}(P_k)$ = ALL SMOOTH CONNECTION 1-FORMS ON P_k

$\mathcal{G}(P_k)$ = GAUGE GROUP

$$= \{ f : P_k \to P_k \;:\; f \text{ IS A DIFFEOMORPHISM}, \; f(p \cdot g) = f(p) \cdot g \;\; \forall p \in P_k \; \forall g \in SU(2), \; \pi_k \circ f = \pi_k \}$$

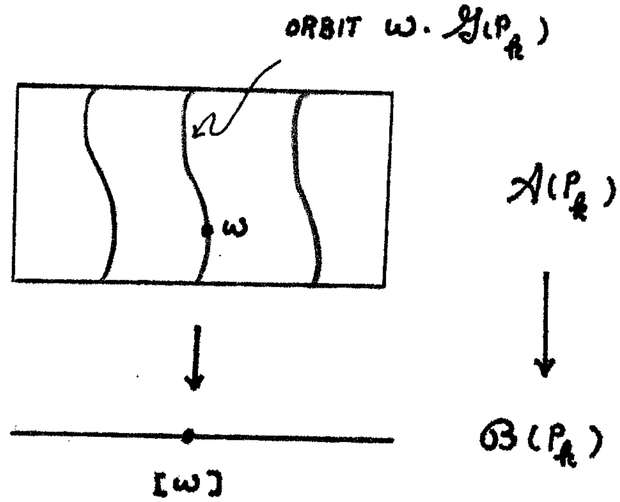
RIGHT ACTION OF $\mathcal{G}(P_k)$ ON $\mathcal{A}(P_k)$: $\quad \omega \cdot f = f^* \omega$

GAUGE EQUIVALENT CONNECTIONS : $\quad \omega_2 = \omega_1 \cdot f$

MODULI SPACE OF GAUGE EQUIVALENCE CLASSES OF CONNECTIONS :

$$\mathcal{B}(P_k) = \mathcal{A}(P_k) / \mathcal{G}(P_k)$$

$$= \{ [\omega] : \omega \in \mathcal{A}(P_k) \}$$



NOTE : FOR THE ANALYSIS THAT FOLLOWS ALL OF THESE C^∞ OBJECTS MUST BE REPLACED BY " APPROPRIATE SOBOLEV COMPLETIONS ". CONSIDER IT DONE !

$\mathcal{A}(P_k)$ IS AN AFFINE SPACE MODELED ON THE (INFINITE-DIMENSIONAL) VECTOR SPACE

$$\Omega^1_{ad}(P_k, su(2)) = \{ \varphi \in \Omega^1(P_k, su(2)) : \varphi \text{ IS HORIZONTAL AND } \sigma_g^* \varphi = g^{-1} \varphi g \}$$

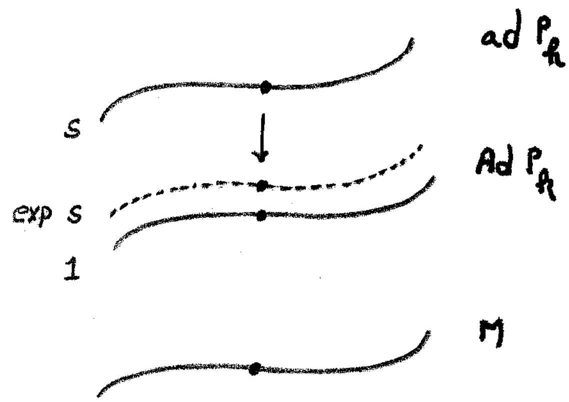
$$\cong \Omega^1(M, ad\, P_k)$$

THUS, $\mathcal{A}(P_k)$ IS AN INFINITE-DIMENSIONAL MANIFOLD WITH TANGENT SPACES

$$T_\omega(\mathcal{A}(P_k)) \cong \Omega^1_{ad}(P_k, su(2)) \cong \Omega^1(M, ad\, P_k)$$

$\mathcal{G}(P_k)$ CAN BE IDENTIFIED WITH THE SPACE OF SECTIONS OF THE NONLINEAR ADJOINT BUNDLE $\mathrm{Ad}\,P_k$ AND IS A HILBERT LIE GROUP WITH LIE ALGEBRA THAT CAN BE IDENTIFIED WITH THE SPACE OF SECTIONS OF THE (ORDINARY) ADJOINT BUNDLE $\mathrm{ad}\,P_k$.

POINTWISE EXPONENTIATE THE LATTER TO GET THE FORMER:



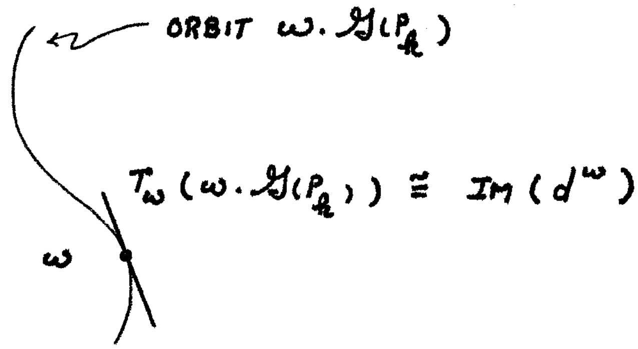
$$s \in \Omega^0(M, \mathrm{ad}\,P_k) \longrightarrow \exp s \in \Omega^0(M, \mathrm{Ad}\,P_k) = \mathcal{G}(P_k)$$

FOR A FIXED $\omega \in \mathcal{A}(P_k)$ THE MAP

$$f \to \omega . f \; : \mathcal{G}(P_k) \to \mathcal{A}(P_k)$$

HAS A DERIVATIVE AT $1 \in \mathcal{G}(P_k)$ THAT CAN BE IDENTIFIED WITH

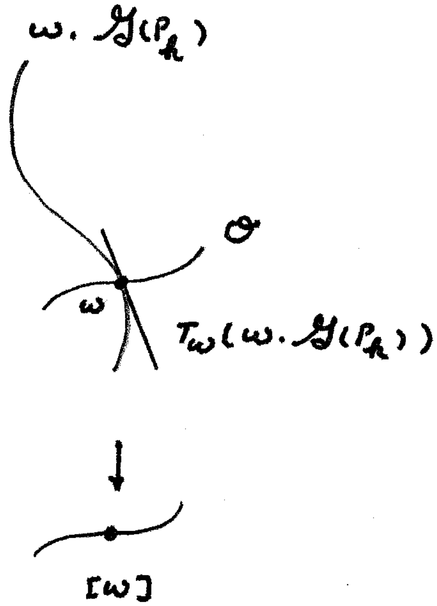
$$-d^\omega : \Omega^0(M, \mathrm{ad}\,P_k) \to \Omega^1(M, \mathrm{ad}\,P_k)$$



WOULD LIKE A "SLICE" OF THE $\mathcal{G}(P_k)$-ACTION ON $\mathcal{A}(P_k)$ NEAR ω, I.E., A SUBMANIFOLD $\mathcal{O}$ OF $\mathcal{A}(P_k)$ SUCH THAT

$$T_\omega(\mathcal{A}(P_k)) \cong T_\omega(\omega.\mathcal{G}(P_k)) \oplus T_\omega(\mathcal{O})$$

AND THE RESTRICTION TO $\mathcal{O}$ OF THE PROJECTION $\mathcal{A}(P_k) \to \mathcal{B}(P_k)$ IS INJECTIVE.



$$\boxed{\text{CHOOSE A RIEMANNIAN METRIC } g \text{ ON } M}$$

TOGETHER WITH AN ad-INVARIANT INNER PRODUCT ON $su(2)$ THIS GIVES NATURAL INNER PRODUCTS ON EACH $\Omega^i(M, \mathrm{ad}\, P_k)$ AND SO A FORMAL ADJOINT

$$\delta^\omega : \Omega^1(M, \mathrm{ad}\, P_k) \to \Omega^0(M, \mathrm{ad}\, P_k)$$

FOR

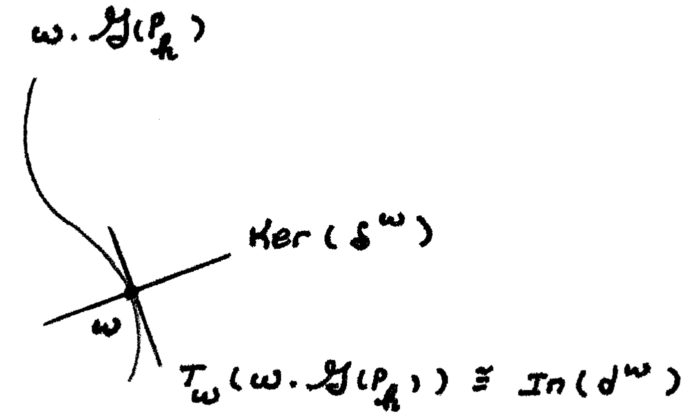
$$d^\omega : \Omega^0(M, \mathrm{ad}\, P_k) \to \Omega^1(M, \mathrm{ad}\, P_k).$$

$$\delta^{\omega} \circ d^{\omega} : \Omega^{0}(M, \mathrm{ad}\, P_k) \longrightarrow \Omega^{0}(M, \mathrm{ad}\, P_k)$$

IS (FORMALLY) SELF-ADJOINT AND ELLIPTIC.

ELLIPTIC THEORY (GENERALIZED HODGE DECOMPOSITION) GIVES THE ORTHOGONAL DECOMPOSITION

$$T_{\omega}(\mathcal{A}(P_k)) \cong \Omega^{1}(M, \mathrm{ad}\, P_k) \cong \mathrm{Im}(d^{\omega}) \oplus \mathrm{Ker}(\delta^{\omega})$$



$\mathrm{Ker}(\delta^{\omega})$ IS THE TANGENT SPACE TO THE AFFINE SUBSPACE

$$\omega + \mathrm{Ker}(\delta^{\omega}).$$

HOWEVER, PROJECTION OF $\omega + \mathrm{Ker}(\delta^{\omega})$ INTO $\mathcal{B}(P_k)$ NEED NOT BE INJECTIVE NEAR ω UNLESS ω IS "IRREDUCIBLE".

<u>STABILIZER</u> (OR <u>ISOTROPY</u> <u>SUBGROUP</u>) OF ω IS

$$\{ f \in \mathcal{G}(P_k) : f^{*}\omega = \omega \} \supseteq \mathbb{Z}_2 .$$

ω IS IRREDUCIBLE IF ITS STABILIZER IS PRECISELY $\mathbb{Z}_2$ AND REDUCIBLE OTHERWISE.

THE FOLLOWING ARE EQUIVALENT:

1. ω IS REDUCIBLE.
2. STABILIZER $(\omega)/\mathbb{Z}_2 \cong U(1)$
3. $d^\omega : \Omega^0(M, \operatorname{ad} P_k) \to \Omega^1(M, \operatorname{ad} P_k)$ HAS NONTRIVIAL KERNEL.

TO SEE THE RELEVANCE OF THIS: THE DERIVATIVE OF

$$\mathcal{G}(P_k) \times (\omega + \operatorname{Ker}(\delta^\omega)) \longrightarrow \mathcal{A}(P_k)$$

$$(f, \omega') \longrightarrow f^*\omega'$$

AT $(1, \omega)$ IS

$$\Omega^0(M, \operatorname{ad} P_k) \oplus \operatorname{Ker}(\delta^\omega) \longrightarrow \Omega^1(M, \operatorname{ad} P_k) \cong \operatorname{Im}(d^\omega) \oplus \operatorname{Ker}(\delta^\omega)$$

$$d^\omega \oplus \operatorname{id}_{\operatorname{Ker}(\delta^\omega)}$$

WHICH IS ALWAYS SURJECTIVE AND INJECTIVE IF ω IS IRREDUCIBLE.

INVERSE FUNCTION THEOREM + BOOTSTRAPPING $\Rightarrow$

LOCALLY INJECTIVE NEAR ω IN $\omega + \operatorname{Ker}(\delta^\omega)$

ω IRREDUCIBLE $\Rightarrow$

A SUFFICIENTLY SMALL NEIGHBORHOOD OF ω IN $\omega + \operatorname{Ker}(\delta^{\omega})$ PROJECTS INJECTIVELY INTO THE MODULI SPACE $\mathcal{B}(P_k)$ AND SO PROVIDES A LOCAL (INFINITE DIMENSIONAL) MANIFOLD STRUCTURE NEAR $[\omega]$.

$\hat{\mathcal{A}}(P_k)$ = IRREDUCIBLE ELEMENTS OF $\mathcal{A}(P_k)$

$$\hat{\mathcal{B}}(P_k) = \hat{\mathcal{A}}(P_k)/\mathcal{G}(P_k)$$

$$T_{[\omega]}(\hat{\mathcal{B}}(P_k)) \cong \operatorname{Ker}(\delta^{\omega})$$

NOW CONSIDER

$Asd(P_k, g)$ = ALL $\omega \in \mathcal{A}(P_k)$ WITH $*F_\omega = -F_\omega$

$\widehat{Asd}(P_k, g)$ = IRREDUCIBLE ELEMENTS OF $Asd(P_k, g)$

$$\mathcal{M}(P_k, g) = Asd(P_k, g)/\mathcal{G}(P_k)$$

$$\hat{\mathcal{M}}(P_k, g) = \widehat{Asd}(P_k, g)/\mathcal{G}(P_k)$$

TO STUDY THE LOCAL STRUCTURE OF THESE MODULI SPACES ASSOCIATE WITH EACH $\omega \in Asd(P_k, g)$ ITS <u>FUNDAMENTAL</u> <u>ELLIPTIC</u> <u>COMPLEX</u> $\mathcal{E}(\omega)$:

$$0 \to \Omega^0(M, ad P_k) \underset{\delta^\omega}{\overset{d^\omega}{\rightleftarrows}} \Omega^1(M, ad P_k) \underset{\delta^\omega_+}{\overset{d^\omega_+}{\rightleftarrows}} \Omega^2_+(M, ad P_k) \to 0$$

WHERE d^ω_+ IS d^ω FOLLOWED BY PROJECTION Pr_+ ONTO THE SELF-DUAL PART.

<u>NOTE</u> : $d^\omega \circ d^\omega = [F_\omega, \cdot\,] \Rightarrow d^\omega_+ \circ d^\omega = [F^+_\omega, \cdot\,]$
WHICH IS ZERO IF ω IS ASD.

ELLIPTIC THEORY $\Rightarrow$ COHOMOLOGY OF $\mathcal{E}(\omega)$ IS FINITE-DIMENSIONAL AND GIVEN BY

$$H^0(\omega) = Ker(d^\omega)$$

$$H^1(\omega) = Ker(d^\omega_+ \,|\, Ker(\delta^\omega))$$

$$H^2(\omega) = Im(d^\omega_+ \,|\, Ker(\delta^\omega))^\perp$$

HERE'S THE SIGNIFICANCE OF THIS :

$$F : \mathcal{A}(P_k) \longrightarrow \Omega^2(M, \mathrm{ad}\, P_k)$$

$$F(\omega) = F_\omega$$

LET $\mathcal{O}_{\omega,\varepsilon}$ = LOCAL SLICE NEAR ω

$$Pr_+ \circ F \mid \mathcal{O}_{\omega,\varepsilon} : \mathcal{O}_{\omega,\varepsilon} \longrightarrow \Omega^2_+(M, \mathrm{ad}\, P_k)$$

$$(Pr_+ \circ F \mid \mathcal{O}_{\omega,\varepsilon})^{-1}(0) = Asd(P_k, g) \cap \mathcal{O}_{\omega,\varepsilon}$$

THE DERIVATIVE OF $Pr_+ \circ F \mid \mathcal{O}_{\omega,\varepsilon}$ IS

$$d_+^\omega \mid Ker(\delta^\omega) : Ker(\delta^\omega) \to \Omega^2_+(M, \mathrm{ad}\, P_k)$$

WHICH IS <u>FREDHOLM</u> (BY FINITE DIMENSIONALITY OF $H^1(\omega)$ AND $H^2(\omega)$) AND IT IS <u>SURJECTIVE</u> IF AND ONLY IF

$$H^2(\omega) = 0.$$

IN THIS CASE THE (BANACH MANIFOLD) <u>IMPLICIT FUNCTION THEOREM</u> GIVES A LOCAL MANIFOLD STRUCTURE FOR $Asd(P_k, g) \cap \mathcal{O}_{\omega,\varepsilon}$ NEAR ω OF DIMENSION

$$\mathrm{DIM}\,\left(Ker(d_+^\omega \mid Ker(\delta^\omega))\right) = \mathrm{DIM}\, H^1(\omega).$$

IF, IN ADDITION,

$$H^0(\omega) = 0$$

(I.E., ω IS IRREDUCIBLE), THEN THE PROJECTION INTO THE MODULI SPACE IS INJECTIVE NEAR ω IN $Asd(P_k, g) \cap \mathcal{O}_{\omega,\varepsilon}$ SO $\hat{\mathcal{M}}(P_k, g)$ HAS A LOCAL MANIFOLD STRUCTURE NEAR $[\omega]$ OF DIMENSION

$$\begin{aligned} \text{DIM } H^1(\omega) &= -\text{DIM } H^0(\omega) + \text{DIM } H^1(\omega) - \text{DIM } H^2(\omega) \\ &= -\text{ INDEX OF THE COMPLEX } \mathcal{E}(\omega) \\ &= 8k - 3(1 + b_2^+(M)) \quad \text{BY THE ATIYAH-SINGER INDEX THEOREM} \end{aligned}$$

<u>NOTE</u>: INDEPENDENT OF ω

BUT "GENERICALLY", $H^0(\omega)$ AND $H^2(\omega)$ ARE ALWAYS TRIVIAL:

<u>GENERIC METRICS THEOREM</u>: LET $\mathcal{R}$ BE THE SPACE OF ALL RIEMANNIAN METRICS ON M. THEN

1. THERE IS A DENSE G_δ-SET IN $\mathcal{R}$ SUCH THAT, FOR EVERY g IN THIS SET, ANY g-ASD CONNECTION ω SATISFIES

$$H^2(\omega) = 0.$$

2. IF $b_2^+(M) > 0$, THEN THERE IS A DENSE G_δ-SET IN $\mathcal{R}$ SUCH THAT, FOR EVERY g IN THIS SET, ANY g-ASD CONNECTION ω SATISFIES $(H^2(\omega) = 0)$ AND

$$H^0(\omega) = 0.$$

IN SHORT, FOR "GENERIC" g,

$$\mathcal{M}(P_k, g) = \hat{\mathcal{M}}(P_k, g)$$

IS (EITHER EMPTY OR) A SMOOTH MANIFOLD OF DIMENSION

$$8k - 3(1 + b_2^+(M)).$$