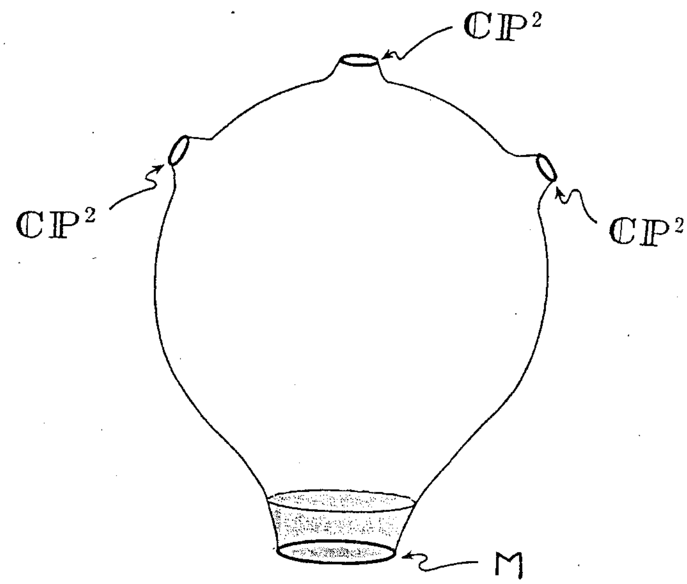
ADDENDUM 3 : "ETC."



HAVING DELETED THE (OPEN) TOP HALF OF EACH CONE AND THE (OPEN) BOTTOM HALF OF THE CYLINDER WE ARE LEFT WITH A MANIFOLD WITH BOUNDARY THAT IS COMPACT (BECAUSE K IS COMPACT) AND ORIENTED WITH BOUNDARY

$$M \sqcup p\,\mathbb{CP}^2 \sqcup q\,\overline{\mathbb{CP}}^2$$

WHERE

$$p + q = m .$$

THUS, M IS COBORDANT TO THE DISJOINT UNION $p\,\mathbb{CP}^2 \sqcup q\,\overline{\mathbb{CP}}^2$ SO

$$\sigma(M) = \sigma(p\,\mathbb{CP}^2 \sqcup q\,\overline{\mathbb{CP}}^2)$$

$$b_2^+(M) - b_2^-(M) = p - q$$

$$b_2^-(M) = q - p$$

$$b_2(M) = q - p \leq q + p = m$$

WE SHOW NEXT THAT $b_2(M) \geq m$:

SELECT $x_1 \in H_2(M;\mathbb{Z})$ WITH $Q_M(x_1, x_1) = -1$ (THERE MUST BE AT LEAST ONE SUCH BECAUSE WE ASSUME THAT $b_2^+(M) = 0$, BUT $b_2(M) \neq 0$).

THEN THERE IS A Q_M - ORTHOGONAL DECOMPOSITION

$$H_2(M;\mathbb{Z}) \cong \mathbb{Z}x_1 \oplus G_1$$

NOW CONSIDER ANY $x_2 \in H_2(M;\mathbb{Z})$ WITH $Q_M(x_2, x_2) = -1$ AND $x_2 \neq \pm x_1$ (IF SUCH A THING HAPPENS TO EXIST). THE SCHWARTZ INEQUALITY GIVES

$$(Q_M(x_1, x_2))^2 < Q_M(x_1, x_1)\, Q_M(x_2, x_2) = 1 .$$

BUT $Q_M(x_1, x_2)$ IS AN INTEGER SO

$$Q_M(x_1, x_2) = 0$$

AND THEREFORE

$$x_2 \in G_1$$

NOW REPEAT THE ARGUMENT INSIDE G_1 AND CONTINUE INDUCTIVELY UNTIL YOU RUN OUT OF $x \in H_2(M;\mathbb{Z})$ FOR WHICH $Q_M(x,x) = -1$ (WHICH YOU WILL BECAUSE $H_2(M;\mathbb{Z})$ IS FINITELY GENERATED).

THE RESULT IS AN ORTHOGONAL DECOMPOSITION

$$H_2(M;\mathbb{Z}) \cong \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_m \oplus G$$

WHERE G IS EITHER EMPTY OR THE ORTHOGONAL COMPLEMENT OF $\mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_m$. IN PARTICULAR,

$$m \le b_2(M).$$

SINCE WE SHOWED EARLIER THAT $b_2(M) \le m$ WE CONCLUDE THAT

$$b_2(M) = m = p+q .$$

THUS, IN FACT, G MUST BE EMPTY AND

$$H_2(M;\mathbb{Z}) \cong \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_m$$

WHERE

$$Q_M(x_i) = -1 , \qquad i = 1, \dots, m$$

AND THE MATRIX OF Q_M RELATIVE TO THE BASIS $\{x_1, \dots, x_m\}$ IS $-\mathrm{id}_{m \times m}$.