ADDENDUM 4 : THE μ-MAP

THE DEFINITION OF THE DONALDSON μ-MAP REQUIRES THE CONSTRUCTION OF A CERTAIN AUXILIARY $SO(3)$-BUNDLE WHICH WE BRIEFLY DESCRIBE AS FOLLOWS : TO ECONOMIZE ON NOTATION WE WILL LET

$$P_k = P, \quad \hat{\mathcal{A}}(P_k) = \hat{\mathcal{A}}, \quad \mathcal{G}(P_k) = \mathcal{G}, \quad \hat{\mathcal{B}}(P_k) = \hat{\mathcal{B}}, \quad \hat{\mathcal{M}}(P_k, g) = \hat{\mathcal{M}}$$

$\mathcal{G}$ ACTS ON $\hat{\mathcal{A}} \times P$ BY

$$(\omega, p) \cdot f = (f^* \omega, f^{-1}(p))$$

AND THE ACTION IS FREE (BECAUSE THE ELEMENTS OF $\hat{\mathcal{A}}$ ARE IRREDUCIBLE).

THE ORBIT SPACE

$$\hat{\mathcal{A}} \times_{\mathcal{G}} P$$

IS A HILBERT MANIFOLD AND

$$\mathcal{G} \hookrightarrow \hat{\mathcal{A}} \times P \longrightarrow \hat{\mathcal{A}} \times_{\mathcal{G}} P$$

$$(\omega, p) \longrightarrow [\omega, p]$$

IS A PRINCIPAL $\mathcal{G}$-BUNDLE.

THERE IS A NATURAL MAP

$$\hat{\mathcal{A}} \times_{\mathcal{G}} P \longrightarrow \hat{\mathcal{B}} \times M$$

$$[\omega, p] \longrightarrow ([\omega], \pi(p))$$

THE ACTION

$$(\omega, p)\cdot g = (\omega, p\cdot g)$$

OF $SU(2)$ ON $\hat{\mathcal{A}} \times P$ COMMUTES WITH THE ACTION OF $\mathcal{G}$ ON $\hat{\mathcal{A}} \times P$ SO IT DESCENDS TO AN ACTION OF $SU(2)$ ON $\hat{\mathcal{A}} \times_{\mathcal{G}} P$:

$$[\omega, p]\cdot g = [\omega, p\cdot g]$$

THIS ACTION IS NOT FREE, BUT BECAUSE THE ELEMENTS OF $\hat{\mathcal{A}}$ ARE IRREDUCIBLE ONE FINDS THAT

$$[\omega, p]\cdot g = [\omega, p] \iff g = \pm 1$$

THUS, THERE IS A FREE

$$SU(2)/\pm 1 \cong SO(3)$$

ACTION ON $\hat{\mathcal{A}} \times_{\mathcal{G}} P$ AND WE HAVE A PRINCIPAL $SO(3)$ - BUNDLE

$$\boxed{\begin{array}{ccl} \mathcal{P} : \quad SO(3) \hookrightarrow & \hat{\mathcal{A}} \times_{\mathcal{G}} P & \\ & \Big\downarrow & [\omega, p] \to ([\omega], \pi(p)) \\ & \hat{\mathcal{B}} \times M & \end{array}}$$

AS DOES ANY $SO(3)$ - BUNDLE, THIS HAS A 1^{ST} PONTRYAGIN CLASS

$$p_1(\mathcal{P}) \in H^4(\hat{\mathcal{B}} \times M; \mathbb{Z})$$

ONE CAN SHOW THAT $p_1(\mathcal{P})$ IS DIVISIBLE BY 4 IN THE SENSE THAT

$$-\tfrac{1}{4}\, p_1(\mathcal{P})$$

IS STILL AN INTEGRAL CLASS.

<u>NOTE</u>: THE REASON FOR INTRODUCING THE $-\frac{1}{4}$ IS AS FOLLOWS: WHEN THE PRINCIPAL $SO(3)$-BUNDLE $\mathcal{P}$ LIFTS TO A PRINCIPAL $SU(2)$-BUNDLE $\mathcal{P}'$ (AS IT ALWAYS DOES WHEN $k = c_2(P)$ IS ODD, FOR EXAMPLE),

$$c_2(\mathcal{P}') = -\tfrac{1}{4}\, p_1(\mathcal{P})$$

NOW, THERE IS A GENERAL OPERATION IN ALGEBRAIC TOPOLOGY CALLED "SLANT PRODUCT". IT IS A MAP

$$H^{p+q}(X \times Y) \times H_p(Y) \longrightarrow H^q(X)$$

$$(\gamma, \alpha) \longrightarrow \gamma/\alpha$$

FIXING $\gamma \in H^{p+q}(X \times Y)$ THIS GIVES A MAP

$$H_p(Y) \longrightarrow H^q(X)$$

$$\alpha \longrightarrow \gamma/\alpha$$

APPLY THIS CONSTRUCTION TO

$$H^4(\hat{\mathcal{B}} \times M; \mathbb{Z}) \times H_2(M;\mathbb{Z}) \longrightarrow H^2(\hat{\mathcal{B}};\mathbb{Z})$$

WITH γ FIXED AT $-\frac{1}{4}p_1(\mathcal{P}) \in H^4(\hat{\mathcal{B}} \times M; \mathbb{Z})$ TO OBTAIN A MAP

$$H_2(M;\mathbb{Z}) \longrightarrow H^2(\hat{\mathcal{B}};\mathbb{Z})$$

$$x \longrightarrow -\tfrac{1}{4}p_1(\mathcal{P})/x$$

AND FOLLOW BY THE RESTRICTION TO $\hat{\mathcal{M}} \subseteq \hat{\mathcal{B}}$ TO OBTAIN

$$\mu : H_2(M;\mathbb{Z}) \to H^2(\hat{\mathcal{M}};\mathbb{Z})$$

TO GET A MORE INTUITIVE FEEL FOR THE μ-MAP WE WILL NOW DESCRIBE IT MORE INFORMALLY IN THE LANGUAGE OF DE RHAM COHOMOLOGY :

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$$H^4(\hat{\mathcal{M}} \times M) \cong \bigoplus_{p+q=4} H^p(\hat{\mathcal{M}}) \otimes H^q(M)$$

SO $-\frac{1}{4}p_1(\mathcal{P}|\hat{\mathcal{M}} \times M)$ RESOLVES INTO A SUM OF $(p,4-p)$-CLASSES. LET $[\alpha_1] \otimes [\alpha_2]$ BE THE "(2,2)-PART". EVERY ELEMENT x OF $H_2(M)$ CAN BE REPRESENTED BY A SMOOTHLY EMBEDDED, ORIENTED SURFACE Σ.

$$\mu(x) = \left(\int_\Sigma [\alpha_2]\right)[\alpha_1]$$