ADDENDUM 5 : DONALDSON POLYNOMIALS

HERE WE WILL ATTEMPT TO PROVIDE A VERY BRIEF MAP OF THE ROAD ONE MUST FOLLOW FROM THE " NAIVE DEFINITION " OF THE DONALDSON INVARIANTS

$$\gamma_{d_k}(M) : H_2(M;\mathbb{Z}) \longrightarrow \mathbb{Z}[\tfrac{1}{2}]$$

$$x \in H_2(M;\mathbb{Z}) \longrightarrow \mu(x) \wedge \overset{d_k}{\cdots} \wedge \mu(x) \in H^{2d_k}(\mathcal{M}(P_k, g);\mathbb{Z})$$

$$\gamma_{d_k}(M)(x) = \int_{\mathcal{M}(P_k, g)} \mu(x) \wedge \overset{d_k}{\cdots} \wedge \mu(x)$$

TO AN HONEST DEFINITION.

FIRST STEP IS THE CONSTRUCTION OF A COMPACTIFICATION

$$\bar{\mathcal{M}}(P_k, g)$$

OF $\mathcal{M}(P_k, g)$.

RELIES ON DEEP ANALYTICAL RESULTS OF KAREN UHLENBECK AND IS ESSENTIALLY AN INTRICATE VARIATION ON THE PHENOMENON WE WITNESSED FOR THE BPST INSTANTONS ON S^4 :

SEQUENCE OF POINTS IN $\mathcal{M}(P_k, g)$ CAN FAIL TO HAVE A CONVERGENT SUBSEQUENCE ONLY IF THE CORRESPONDING CURVATURES HAVE POINTWISE NORMS THAT BECOME INCREASINGLY CONCENTRATED AT A POINT (OR FINITE SET OF POINTS) IN M.

INTUITIVELY, THESE CONVERGE TO "δ - CONNECTIONS" WHICH, BECAUSE OF THEIR SINGULAR NATURE, DO NOT APPEAR IN THE MODULI SPACE.

CONSTRUCTION OF $\bar{\mathcal{M}}(P_k, g)$ OCCURS IN LAYERS ("STRATA") BY ADDING ON THESE "VIRTUAL" CONNECTIONS, THEN LIMITS OF VIRTUAL CONNECTIONS, ...

NO RESTRICTION ON $b_2^+(M)$ REQUIRED. WHEN $b_2^+(M) = 0$ AND $k = 1$ THERE IS ONLY ONE STRATUM AND $\bar{\mathcal{M}}(P_1, g)$ IS OBTAINED FROM OUR EARLIER PICTURE OF $\mathcal{M}(P_1, g)$ BY ATTACHING THE "BOTTOM" COPY $M \times \{0\}$ OF M.

DONALDSON SHOWS THAT THE μ-MAP $\mu : H_2(M; \mathbb{Z}) \rightarrow H^2(\mathcal{M}(P_k, g); \mathbb{Z})$ EXTENDS TO

$$\bar{\mu} : H_2(M; \mathbb{Z}) \longrightarrow H^2(\bar{\mathcal{M}}(P_k, g); \mathbb{Z}).$$

0TH APPROXIMATION : $\gamma_{d_k}(x) = \int_{\mathfrak{M}(P_k,g)} \overbrace{\mu(x)\wedge\cdots\wedge\mu(x)}^{d_k}$

PROBLEM : $\mathfrak{M}(P_k, g)$ IS NOT COMPACT

1ST APPROXIMATION : $\gamma_{d_k}(x) = \int_{\bar{\mathfrak{M}}(P_k,g)} \overbrace{\bar{\mu}(x)\wedge\cdots\wedge\bar{\mu}(x)}^{d_k}$

PROBLEM : $\bar{\mathfrak{M}}(P_k, g)$ IS NOT A MANIFOLD

2ND APPROXIMATION : $\gamma_{d_k}(x) = \langle \overbrace{\bar{\mu}(x)\smile\cdots\smile\bar{\mu}(x)}^{d_k}, [\bar{\mathfrak{M}}(P_k,g)] \rangle$

PROBLEM : $\bar{\mathfrak{M}}(P_k, g)$ ADMITS A FUNDAMENTAL CLASS $[\bar{\mathfrak{M}}(P_k,g)]$ ONLY IF k IS IN THE <u>STABLE RANGE</u>

$$k > \frac{3}{4}(1 + b_2^+(M))$$

OR, EQUIVALENTLY,

$$d_k > \frac{3}{2}(1 + b_2^+(M)).$$

AT LEAST FOR INTEGERS

$$d \equiv -\frac{3}{2}(1 + b_2^+(M)) \bmod 4$$

$$d > \frac{3}{2}(1 + b_2^+(M))$$

WE HAVE A

$$\gamma_d(M) : H_2(M; \mathbb{Z}) \to \mathbb{Z}.$$

TO REMOVE THE RESTRICTIONS ON d :

EXTEND $\gamma_d(M) : H_2(M;\mathbb{Z}) \to \mathbb{Z}$ TO A d-MULTILINEAR MAP

$$\gamma_d(M) : H_2(M;\mathbb{Z}) \times \overset{d}{\cdots} \times H_2(M;\mathbb{Z}) \to \mathbb{Z}$$

$$\gamma_d(M)(x_1,\dots,x_d) = \langle \bar{\mu}(x_1) \smile \cdots \smile \bar{\mu}(x_d),\ [\bar{\mathcal{M}}(P_k,g)] \rangle$$

EXTEND THE DEFINITION TO INCLUDE ARGUMENTS IN $H_0(M;\mathbb{Z})$ WITH ANOTHER μ-MAP

$$\mu : H_0(M;\mathbb{Z}) \longrightarrow H^4(\mathcal{M}(P_k,g);\mathbb{Z})$$

(ANALOGOUS DEFINITION).

$$H_0(M;\mathbb{Z}) \cong \mathbb{Z}$$

GENERATOR η

PROBLEM : $\mu(\eta)$ DOES NOT EXTEND TO THE ENTIRE UHLENBECK COMPACTIFICATION

$\mu(\eta)$ DOES, HOWEVER, ADMIT AN EXTENSION $\bar{\mu}(\eta)$ TO A LARGE ENOUGH SUBSET OF $\bar{\mathcal{M}}(P_k,g)$ THAT, UNDER CERTAIN ADDITIONAL RESTRICTIONS ON d, ONE CAN PRODUCE THE DESIRED EXTENSION OF $\gamma_d(M)$ TO INCLUDE THIS 4-DIMENSIONAL CLASS :

CONSIDER A $d \equiv -\frac{3}{2}(1+b_2^+(M)) \bmod 4$ AND

$$d = a + 2b$$

WHERE $b \geq 0$ AND

$$a > \frac{3}{2}(1+b_2^+(M)).$$

THEN ONE CAN DEFINE

$$\gamma_d(M) : H_2(M;\mathbb{Z}) \times \overset{a}{\cdots} \times H_2(M;\mathbb{Z}) \times H_0(M;\mathbb{Z}) \times \overset{b}{\cdots} \times H_0(M;\mathbb{Z}) \to \mathbb{Z}$$

BY

$$\gamma_d(M)(x_1,\dots,x_a, n_1\eta,\dots,n_b\eta) =$$

$$n_1\cdots n_b \langle \bar{\mu}(x_1) \smile \cdots \smile \bar{\mu}(x_a) \smile \bar{\mu}(\eta)^b, [\bar{\mathcal{M}}(P_k,g)] \rangle$$

WHEN $\mathcal{M}(P_k,g) \neq \phi$ AND $\gamma_d(M) \equiv 0$ WHEN $\mathcal{M}(P_k,g) = \phi$ (HERE k IS THE INTEGER FOR WHICH $2d = 8k - 3(1+b_2^+(M))$).

FOR ANY INTEGER $k > 0$ SAY THAT A SEQUENCE

$$S = (x_1,\dots,x_a, n_1\eta,\dots,n_b\eta)$$

WITH $x_i \in H_2(M;\mathbb{Z})$, $i = 1,\dots,a$, AND $n_j\eta \in H_0(M;\mathbb{Z})$, $j = 1,\dots,b$, IS <u>k-STABLE FOR</u> M IF $b \geq 0$, $a > \frac{3}{2}(1+b_2^+(M))$, AND $a + 2b = d = 4k - \frac{3}{2}(1+b_2^+(M))$.

AT THIS POINT WE HAVE DEFINED $\gamma_d(M)(S)$ WHENEVER S IS k- STABLE FOR M.

NEXT WE NEED A "BLOW-UP FORMULA" OF DONALDSON.

<u>RECALL</u>:

- BLOW-UP OF M : $M \# \overline{\mathbb{CP}}^2$
- $H_2(\overline{\mathbb{CP}}^2; \mathbb{Z}) \cong \mathbb{Z}$, $Q_{\overline{\mathbb{CP}}^2} = (-1)$, $b_2^+(\overline{\mathbb{CP}}^2) = 0$
- $H_2(M \# \overline{\mathbb{CP}}^2; \mathbb{Z}) \cong H_2(M; \mathbb{Z}) \oplus H_2(\overline{\mathbb{CP}}^2; \mathbb{Z})$
- $H_+^2(\overline{\mathbb{CP}}^2, \mathbb{R}) \cong 0$
- $H_+^2(M \# \overline{\mathbb{CP}}^2; \mathbb{R}) \cong H_+^2(M; \mathbb{R})$

THUS,

ORIENTING $H_+^2(M; \mathbb{R})$ ORIENTS $H_+^2(M \# \overline{\mathbb{CP}}^2; \mathbb{R})$

AND

STABLE RANGES OF M AND $M \# \overline{\mathbb{CP}}^2$ ARE THE SAME.

FIX A GENERATOR e FOR $H_2(\overline{\mathbb{CP}}^2; \mathbb{Z}) \hookrightarrow H_2(M \# \overline{\mathbb{CP}}^2; \mathbb{Z})$.

THEN

1. S k-STABLE FOR M (AND SO ALSO FOR $M \# \overline{\mathbb{CP}}^2$) $\Rightarrow$

$$\gamma_{d_k}(M \# \overline{\mathbb{CP}}^2)(S) = \gamma_{d_k}(M)(S).$$

2. SUPPOSE $i = 1, 2,$ OR 3 AND S IS NOT k-STABLE FOR M, BUT $(S, e, \overset{i}{\ldots}, e)$ IS k-STABLE FOR $M \# \overline{\mathbb{CP}}^2$. THEN

$$\gamma_{d_k}(M \# \overline{\mathbb{CP}}^2)(S, e, \overset{i}{\ldots}, e) = 0.$$

3. S k-STABLE FOR M (AND SO ALSO FOR $M \# \overline{\mathbb{CP}}^2$) $\Rightarrow$ (S, e, e, e, e) $(k+1)$-STABLE FOR $M \# \overline{\mathbb{CP}}^2$ AND

$$\gamma_{d_k}(M)(S) = -\tfrac{1}{2}\, \gamma_{d_{k+1}}(M \# \overline{\mathbb{CP}}^2)(S, e, e, e, e).$$

FROM THESE:

$$M \# n\overline{\mathbb{CP}}^2 = M \# \overline{\mathbb{CP}}^2_1 \# \cdots \# \overline{\mathbb{CP}}^2_n$$

e_i GENERATOR FOR $H_2(\overline{\mathbb{CP}}^2_i; \mathbb{Z})$

S k-STABLE FOR M $\Rightarrow$ $(S, e_1, e_1, e_1, e_1, \ldots, e_n, e_n, e_n, e_n)$ $(k+n)$-STABLE FOR $M \# n\overline{\mathbb{CP}}^2$ AND

$$\boxed{\gamma_{d_k}(M)(S) = (-\tfrac{1}{2})^n\, \gamma_{d_{n+k}}(M \# n\overline{\mathbb{CP}}^2)(S, e_1, e_1, e_1, e_1, \ldots, e_n, e_n, e_n, e_n)}$$

HERE'S THE POINT TO THIS BLOW-UP FORMULA:

EVEN WHEN $S = (x_1, \dots, x_a, n_1\eta, \dots, n_b\eta)$ SATISFIES

$$a + 2b = d_k$$

BUT NOT

$$a > \tfrac{3}{2}(1 + b_2^+(M))$$

SO THAT S IS NOT k-STABLE FOR M AND $\gamma_{d_k}(M)(S)$ HAS NOT YET BEEN DEFINED, THE RIGHT-HAND SIDE WILL BE DEFINED PROVIDED ONLY THAT n IS SUFFICIENTLY LARGE.

MOREOVER, THE RIGHT-HAND SIDE TAKES THE SAME VALUE FOR ALL SUFFICIENTLY LARGE n (BY #3 ABOVE) SO WE MAY USE IT TO DEFINE THE LEFT-HAND SIDE FOR ANY S WITH $a + 2b = d_k$.

$\gamma_{d_k}(M)$ NOW TAKES VALUES IN $\mathbb{Z}[\tfrac{1}{2}]$.

STATUS REPORT: LET d BE AN INTEGER SATISFYING

$$d \equiv -\tfrac{3}{2}(1 + b_2^+(M)) \bmod 4 .$$

CHOOSE k SO THAT

$$2d = 8k - 3(1 + b_2^+(M))$$

AND CONSIDER

$$SU(2) \hookrightarrow P_k \longrightarrow M$$

AND

$$\mathcal{M}(P_k, g)$$

WHERE g IS GENERIC. THE FORMAL DIMENSION OF $\mathcal{M}(P_k, g)$ IS $2d$.

- $d < 0 \Rightarrow \gamma_d(M)$ IS TAKEN TO BE IDENTICALLY ZERO
- $d = 0 \Rightarrow$ WE HAVE DESCRIBED (PREVIOUS LECTURE) A NUMERICAL INVARIANT
$$\gamma_0(M) \in \mathbb{Z}.$$
- $d > 0 \Rightarrow$ WE HAVE, FOR ALL NON-NEGATIVE INTEGERS a AND b WITH $a + 2b = d$, A MULTILINEAR MAP

$$\gamma_d(M) : H_2(M;\mathbb{Z}) \times \overset{a}{\cdots} \times H_2(M;\mathbb{Z}) \times H_0(M;\mathbb{Z}) \times \overset{b}{\cdots} \times H_0(M;\mathbb{Z}) \to \mathbb{Z}[\tfrac{1}{2}]$$

DEFINED, IF $a > \frac{3}{2}(1 + b_2^+(M))$, BY

$$\gamma_d(M)(x_1, \ldots, x_a, n_1\eta, \ldots, n_b\eta) =$$

$$n_1 \cdots n_b \langle \bar{\mu}(x_1) \smile \cdots \smile \bar{\mu}(x_a) \smile \bar{\mu}(\eta)^b, [\bar{\mathcal{M}}(P_k, g)] \rangle$$

AND BY THE BLOW-UP FORMULA WITH n SUFFICIENTLY LARGE OTHERWISE.

NEXT WE RELAX THE REQUIREMENT

$$d \equiv -\tfrac{3}{2}(1+b_2^+(M)) \bmod 4$$

TO

$$d \equiv -\tfrac{3}{2}(1+b_2^+(M)) \bmod 2 .$$

IF $d > 0$ SATISFIES THE MOD 2 CONGRUENCE, BUT <u>NOT</u> THE MOD 4 CONGRUENCE, THEN

$$d+2 \equiv -\tfrac{3}{2}(1+b_2^+(M)) \bmod 4$$

SO

$$\gamma_{d+2}(M) : H_2(M;\mathbb{Z}) \times \overset{d}{\cdots} \times H_2(M;\mathbb{Z}) \times H_0(M;\mathbb{Z}) \to \mathbb{Z}[\tfrac{1}{2}]$$

IS DEFINED AND WE CAN DEFINE

$$\gamma_d(M) : H_2(M;\mathbb{Z}) \times \overset{d}{\cdots} \times H_2(M;\mathbb{Z}) \to \mathbb{Z}[\tfrac{1}{2}]$$

BY

$$\gamma_d(M)(x_1,\dots,x_d) = \tfrac{1}{2}\gamma_{d+2}(M)(x_1,\dots,x_d,\eta) .$$

FINALLY, IF $d \not\equiv -\tfrac{3}{2}(1+b_2^+(M)) \bmod 2$, THEN WE TAKE

$$\gamma_d(M) : H_2(M;\mathbb{Z}) \times \cdots \times H_2(M;\mathbb{Z}) \to \mathbb{Z}[\tfrac{1}{2}]$$

TO BE IDENTICALLY ZERO.

WRITING

$$\gamma_d(M)(\overset{d}{x,\dots,x}) = \gamma_d(M)(x)$$

WE HAVE THE FULL CONTINGENT OF DONALDSON INVARIANTS

$$\gamma_d(M) : H_2(M;\mathbb{Z}) \to \mathbb{Z}[\tfrac{1}{2}]$$

$$d = 1, 2, \ldots$$

(TOGETHER WITH $\gamma_0(M) \in \mathbb{Z}$).

FROM THESE KRONHEIMER AND MROWKA BUILT THE DONALDSON SERIES

$$\mathcal{D}_M(x) = \sum_{d=0}^{\infty} \frac{\gamma_d(M)(x)}{d!}$$

$$= \sum \frac{\gamma_d(M)(\overset{d}{x,\ldots,x})}{d!} + \frac{1}{2}\sum \frac{\gamma_{d+2}(\overset{d}{x,\ldots,x},\eta)}{d!}$$

(MOD 4 CONGRUENCE) (MOD 2, BUT NOT MOD 4 CONGRUENCE)

M IS OF <u>D-SIMPLE TYPE</u> IF

$$\gamma_{d+4}(M)(x_1,\ldots,x_d,\eta,\eta) = 4\,\gamma_d(M)(x_1,\ldots,x_d)$$

FOR ALL $d > 0$ AND ALL $x_1,\ldots,x_d \in H_2(M;\mathbb{Z})$.

FOR THESE ONE CAN

- INDUCTIVELY EXTRACT ALL OF THE

$$\gamma_d(M)(x_1,\dots,x_a, n_1\eta,\dots,n_b\eta)$$

$(a + 2b = d)$ FROM $\mathcal{D}_M(x)$.

- PROVE THE KRONHEIMER - MROWKA STRUCTURE THEOREM (STATED IN THE PREVIOUS LECTURE).

- FORMULATE WITTEN'S CONJECTURE (STILL TO COME).