ADDENDUM 6

THE EULER CLASS OF TS^2

CONSIDER THE 2-SPHERE S^2 WITH ITS USUAL ORIENTATION AND RIEMANNIAN METRIC AND LET $TS^2 \xrightarrow{\pi} S^2$ BE ITS TANGENT BUNDLE. THE CORRESPONDING ORIENTED, ORTHONORMAL FRAME BUNDLE IS

$$SO(2) \hookrightarrow F_{SO}(TS^2) \xrightarrow{\pi_{SO}} S^2.$$

A CONNECTION ω ON $F_{SO}(TS^2)$ IS AN $\mathfrak{so}(2)$-VALUED 1-FORM

$$\omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} = \begin{pmatrix} 0 & \omega^1 \\ -\omega^1 & 0 \end{pmatrix} = \omega^1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

ON $F_{SO}(TS^2)$. SINCE $SO(2)$ IS ABELIAN, THE CURVATURE IS

$$\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} = d\omega = d\omega^1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

THE LEVI-CIVITA CONNECTION ON $F_{SO}(TS^2)$ IS CHARACTERIZED BY

$$d\theta^i = -\omega_{ij} \wedge \theta^j$$

WHERE $\{\theta^1, \theta^2\}$ IS THE BASIS OF 1-FORMS DUAL TO AN ORIENTED, ORTHONORMAL FRAME FIELD $\{e_1, e_2\}$ ON S^2 (I.E., A SECTION OF $F_{SO}(TS^2)$). INSERT IF ϕ AND θ ARE THE USUAL SPHERICAL COORDINATES ON S^2, THEN WE MAY TAKE $\{e_1, e_2\} = \{\frac{\partial}{\partial\phi}, \frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\}$ AND $\{\theta^1, \theta^2\} = \{d\phi, \sin\phi\, d\theta\}$. SINCE

INSERT: THE SAME EQUATIONS ARE CLEARLY ALSO ~~SATISFIED BY THE~~ PULLBACKS OF THESE ~~FORMS TO S^2 BY A~~ SECTION OF $F_{SO}(TS^2)$ AND WE WILL USE THE SAME SYMBOLS TO DENOTE THESE PULLBACKS.

$$d\theta^1 = 0 = -\omega_{11} \wedge \theta^1 - \omega_{12} \wedge \theta^2 = -\omega_{12} \wedge \theta^2 = -\omega_{12} \wedge (\sin\phi\, d\theta)$$

$$\begin{aligned}
d\theta^2 = d(\sin\phi\, d\theta) &= \cos\phi\, d\phi \wedge d\theta \\
&= -\omega_{21} \wedge \theta^1 - \omega_{22} \wedge \theta^2 \\
&= -\omega_{21} \wedge \theta^1 \\
&= \omega_{12} \wedge \theta^1 \\
&= \omega_{12} \wedge d\phi
\end{aligned}$$

$$-\cos\phi\, d\theta \wedge d\phi = \omega_{12} \wedge d\phi$$

WE HAVE

$$\omega_{12} = -\cos\phi \, d\theta$$

(WHICH ALSO SATISFIES THE FIRST CONDITION). THUS,

$$\omega = \begin{pmatrix} 0 & -\cos\phi\, d\theta \\ \cos\phi\, d\theta & 0 \end{pmatrix} = -\cos\phi\, d\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

AND

$$\Omega = \sin\phi\, d\phi \wedge d\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sin\phi\, d\phi \wedge d\theta \\ -\sin\phi\, d\phi \wedge d\theta & 0 \end{pmatrix}$$

THUS,

$$(2\pi)^{-1} \, \mathrm{PFAF}(\Omega) = \frac{1}{4\pi} \sum_{\sigma \in S_2} (-1)^{\sigma} \Omega_{\sigma(1)\sigma(2)}$$

$$= \frac{1}{4\pi} \left[\Omega_{12} - \Omega_{21} \right] = \frac{1}{2\pi} \Omega_{12}$$

$$= \frac{1}{2\pi} \sin\phi \, d\phi \wedge d\theta$$

THEN, AS EXPECTED,

$$\int_{S^2} \frac{1}{2\pi} \sin\phi \, d\phi \wedge d\theta = 2 = \chi(S^2)$$