SEIBERG - WITTEN THEORY : PROLOGUE

WITTEN'S 1988 TOPOLOGICAL QUANTUM FIELD THEORY (TQFT) :

M AS BEFORE WITH RIEMANNIAN METRIC g. FIX SOME

$$SU(2) \hookrightarrow P \xrightarrow{\pi} M$$

FIELD CONTENT :

GAUGE FIELD (CONNECTION) ω WITH CURVATURE $F_\omega \in \Omega^2(M, \mathrm{ad}P)$

+ " MATTER FIELDS "

BOSONIC	FERMIONIC
$\phi \in \Omega^0(M, \mathrm{ad}P)$	$\eta \in \Omega^0(M, \mathrm{ad}P)$
$\lambda \in \Omega^0(M, \mathrm{ad}P)$	$\psi \in \Omega^1(M, \mathrm{ad}P)$
	$\chi \in \Omega^2_+(M, \mathrm{ad}P)$

$$\Phi = (\omega, \phi, \lambda, \eta, \psi, \chi)$$

+ " GHOST NUMBERS " + " SUPERSYMMETRY OPERATOR "

+ DONALDSON - WITTEN ACTION $S_{DW}[\Phi]$

$$S_{DW}[\Phi] = \int_M Tr\{ \tfrac{1}{4} F_\omega \wedge {}^*F_\omega + \tfrac{1}{4} F_\omega \wedge F_\omega - \tfrac{1}{2}\psi \wedge [\phi, \psi]$$
$$- i\, d^\omega \zeta \wedge \psi - 2i[\zeta, {}^*\zeta]\lambda + i\phi d^{\omega *} d^\omega \lambda$$
$$- \zeta \wedge {}^* d^\omega \eta \}$$

<u>PARTITION</u> <u>FUNCTION</u>: $Z_{DW} = \int e^{-S_{DW}[\Phi]/e^2} \mathcal{D}\Phi$

<u>EXPECTATION</u> <u>VALUES</u> OF <u>OBSERVABLES</u> $\mathcal{O}$:

$$\langle \mathcal{O} \rangle = \int e^{-S_{DW}[\Phi]/e^2} \mathcal{O}[\Phi]\, \mathcal{D}\Phi$$

SYMMETRIES BUILT INTO S_{DW} + FORMAL PATH INTEGRAL MANIPULATIONS "PROVE" THAT THESE ARE INDEPENDENT OF THE CHOICES OF g AND e.

FORMALLY COMPUTING Z_{DW} IN THE "WEAK COUPLING LIMIT" $e \to 0$, THE STATIONARY PHASE APPROXIMATION HAPPENS TO BE EXACT AND THE INTEGRAL LOCALIZES TO THE ANTI-SELF-DUAL MODULI SPACE GIVING, WHEN $8k - 3(1 + b_2^+(M)) = 0$, THE 0-DIMENSIONAL DONALDSON INVARIANT.

$$Z_{DW} = \gamma_0(M)$$

WITTEN SIMILARLY OBTAINS THE REMAINING DONALDSON INVARIANTS (FORMALLY) AS EXPECTATION VALUES FOR A FAMILY OF OBSERVABLES PARAMETRIZED BY $x \in H_2(M;\mathbb{Z})$.

<u>DUALITY</u> <u>IN</u> <u>WITTEN'S</u> <u>TQFT</u> : 1988 - SPRING, 1994

$e \to 0$	$e \to \infty$
WEAK COUPLING	STRONG COUPLING
ULTRAVIOLET	INFRARED
NON-ABELIAN	ABELIAN
PERTURBATIVE	NONPERTURBATIVE
COMPUTABLE	INTRACTIBLE
$\boxed{\text{DONALDSON INVARIANTS}}$	$\boxed{\quad ? \quad}$

SEIBERG-WITTEN (FALL, 1994) : EXACT SOLUTIONS IN THE INFRARED

$\boxed{\text{SEIBERG-WITTEN INVARIANTS}}$

THESE SEIBERG-WITTEN INVARIANTS ARISE IN MUCH THE SAME WAY AS THE DONALDSON INVARIANTS FROM MODULI SPACES OF SOLUTIONS TO THE "SEIBERG-WITTEN EQUATIONS".

ALGEBRAIC PRELIMINARIES (MORE DETAILS IN ADDENDUM 14)

MATRIX MODEL OF THE (REAL) ALGEBRA $\mathbb{H}$ OF QUATERNIONS:

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$q = q^1 \mathbb{1} + q^2 I + q^3 J + q^4 K$$

$$\bar{q} = q^1 \mathbb{1} - q^2 I - q^3 J - q^4 K$$

= CONJUGATE TRANSPOSE OF THE MATRIX q

MATRIX MODEL OF $\mathbb{R}^4$: ALL 4×4 COMPLEX MATRICES OF THE FORM

$$x = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}$$

WITH

$$\|x\|^2 = \det x$$

$$\langle x, y \rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 \right).$$

ORTHONORMAL BASIS FOR $\mathbb{R}^4$:

$$E_1 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}$$

SATISFIES

$$E_i E_j + E_j E_i = -2\langle E_i, E_j \rangle \mathbb{1}, \quad i, j = 1,2,3,4$$

REAL SUBALGEBRA OF $\mathbb{C}^{4\times 4}$ GENERATED BY $\{E_1, E_2, E_3, E_4\}$ IS THE <u>REAL CLIFFORD ALGEBRA</u> OF $\mathbb{R}^4$ AND IS DENOTED $Cl(\mathbb{R}^4)$.

COMPLEX SUBALGEBRA OF $\mathbb{C}^{4\times 4}$ GENERATED BY $\{E_1, E_2, E_3, E_4\}$ IS THE <u>COMPLEX CLIFFORD ALGEBRA</u> OF $\mathbb{R}^4$ AND IS DENOTED $Cl(\mathbb{R}^4) \otimes \mathbb{C}$.

BASIS :

$$E_0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = \mathbb{1}$$

$$E_1 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}$$

$$E_1E_2 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad E_1E_3 = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}, \quad E_1E_4 = \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix}$$

$$E_2E_3 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \quad E_2E_4 = \begin{pmatrix} -J & 0 \\ 0 & -J \end{pmatrix}, \quad E_3E_4 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$E_1E_2E_3 = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}, \quad E_1E_2E_4 = \begin{pmatrix} 0 & -J \\ J & 0 \end{pmatrix}$$

$$E_1E_3E_4 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad E_2E_3E_4 = \begin{pmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

$$E_1E_2E_3E_4 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

$\mathbb{Z}_2$ - GRADINGS :

$$Cl(\mathbb{R}^4) \cong Cl_0(\mathbb{R}^4) \oplus Cl_1(\mathbb{R}^4)$$

DIAGONAL $\oplus$ ANTI-DIAGONAL

EVEN $\oplus$ ODD

AND SIMILARLY FOR $Cl(\mathbb{R}^4) \otimes \mathbb{C}$

CENTERS :

$$Z(Cl(\mathbb{R}^4)) = \mathrm{SPAN}_{\mathbb{R}}\{E_0\} \cong \mathbb{R}$$

$$Z(Cl(\mathbb{R}^4) \otimes \mathbb{C}) = \mathrm{SPAN}_{\mathbb{C}}\{E_0\} \cong \mathbb{C}$$

IDENTIFY $U(1)$ WITH THE SUBSET

$$U(1) = \{ e^{\theta i} E_0 : \theta \in \mathbb{R} \}$$

OF $Cl(\mathbb{R}^4) \otimes \mathbb{C}$ (AND GENERALLY WRITE $e^{\theta i}$ FOR $e^{\theta i} E_0 = e^{\theta i}\mathbb{1}$).

IDENTIFY $\mathbb{R}^4 = \text{SPAN}_{\mathbb{R}} \{E_1, E_2, E_3, E_4\} \subseteq Cl(\mathbb{R}^4) \subseteq Cl(\mathbb{R}^4) \otimes \mathbb{C}$.

$$xy + yx = -2\langle x, y\rangle \mathbb{1} \qquad \forall\, x, y \in \mathbb{R}^4$$

$Cl^{\times}(\mathbb{R}^4)$ = MULTIPLICATIVE GROUP OF UNITS IN $Cl(\mathbb{R}^4)$

$x \in \mathbb{R}^4$ WITH $\|x\| = 1 \Rightarrow x \in Cl^{\times}(\mathbb{R}^4)$ AND $x^{-1} = -x$

$\text{PIN}(\mathbb{R}^4)$ = SUBGROUP OF $Cl^{\times}(\mathbb{R}^4)$ GENERATED BY $\{x \in \mathbb{R}^4 : \|x\| = 1\} = S^3$

$$\text{SPIN}(\mathbb{R}^4) = \text{PIN}(\mathbb{R}^4) \cap Cl_0(\mathbb{R}^4)$$

(USUALLY WRITTEN $\text{SPIN}(4)$)

<u>THEOREM</u> : $\text{SPIN}(4) = \left\{ \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} : U_1, U_2 \in SU(2) \right\}$ IS THE UNIVERSAL DOUBLE COVER OF $SO(4)$.

$$\text{SPIN} : \text{SPIN}(4) \longrightarrow SO(\mathbb{R}^4) \cong SO(4)$$

$$(\text{SPIN}(u))(x) = u x u^{-1}$$

COMPLEX ANALOGUE OF $\text{SPIN}(4)$:

$\text{SPIN}^c(4)$ = SUBGROUP OF $Cl^{\times}(\mathbb{R}^4) \otimes \mathbb{C}$ GENERATED BY $\text{SPIN}(4)$ AND $U(1)$.

$$= \left\{ e^{\theta i} \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} : \theta \in \mathbb{R},\ U_1, U_2 \in SU(2) \right\}$$

$$= \left\{ \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix} : U_\pm \in U(2),\ \det U_+ = \det U_- \right\}$$

SPIN(4) AND SPIN^c(4) ARE COMPACT LIE GROUPS.

$$\mathrm{SPIN}(4) \times U(1) \longrightarrow \mathrm{SPIN}^c(4)$$

$$(\mu, e^{\theta i}) \longrightarrow e^{\theta i}\mu$$

SURJECTIVE HOMOMORPHISM WITH KERNEL $\mathbb{Z}_2 = \pm(\mathbb{1}, \mathbb{1})$

$$\mathrm{SPIN}^c(4) \cong \mathrm{SPIN}(4) \times U(1)/\mathbb{Z}_2$$

SOME MAPPINGS:

1. $\boxed{\delta : \mathrm{SPIN}^c(4) \rightarrow U(1)}$

$$\delta(\xi) = \delta\begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix} = \delta\begin{pmatrix} e^{\theta i}U_1 & 0 \\ 0 & e^{\theta i}U_2 \end{pmatrix}$$

$$= \det U_+ = \det U_- = e^{2\theta i}$$

SURJECTIVE HOMOMORPHISM WITH KERNEL SPIN(4)

2. $\boxed{\pi : \mathrm{SPIN}^c(4) \longrightarrow SO(4)}$

$$\pi(\xi) = \pi(e^{\theta i}\mu) = ad_\xi = ad_\mu \in SO(\mathbb{R}^4) \cong SO(4)$$

3. $\boxed{\mathrm{SPIN}^c : \mathrm{SPIN}^c(4) \longrightarrow SO(4) \times U(1)}$

$$\mathrm{SPIN}^c(\xi) = \mathrm{SPIN}^c(e^{\theta i}\mu) = (\pi(\xi), \delta(\xi))$$

$$= (\mathrm{SPIN}(\mu), e^{2\theta i})$$

SURJECTIVE HOMOMORPHISM WITH KERNEL $\mathbb{Z}_2 = \pm \mathbb{1}$

LIE ALGEBRAS OF SPIN(4) AND SPIN^c(4):

$$spin(4) \cong su(2) \oplus su(2) \equiv \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} : A_1, A_2 \in su(2) \right\}$$

$$spin^c(4) \cong spin(4) \oplus u(1) \equiv \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} + ti \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : A_1, A_2 \in su(2),\ t \in \mathbb{R} \right\}$$

$Cl(\mathbb{R}^4) \otimes \mathbb{C} \subseteq \mathbb{C}^{4\times 4}$ AND $\dim_{\mathbb{C}}(Cl(\mathbb{R}^4) \otimes \mathbb{C}) = 16 \Rightarrow$

$$Cl(\mathbb{R}^4) \otimes \mathbb{C} = \mathbb{C}^{4\times 4} \equiv END_{\mathbb{C}}(S_{\mathbb{C}})$$

$$S_{\mathbb{C}} = \mathbb{C}^4 \quad \text{WITH} \quad \langle z, w \rangle = \bar{z}^1 w^1 + \bar{z}^2 w^2 + \bar{z}^3 w^3 + \bar{z}^4 w^4$$

THE ELEMENTS OF $Cl(\mathbb{R}^4) \otimes \mathbb{C}$ (IN PARTICULAR, THOSE IN $\mathbb{R}^4$, $Cl(\mathbb{R}^4)$, SPIN(4) AND SPIN^c(4)) ALL ACT ON $S_{\mathbb{C}}$.

THIS ACTION IS CALLED CLIFFORD MULTIPLICATION

AND IS DENOTED WITH A DOT $\cdot$

RESULTING REPRESENTATIONS:

1. $Cl(\mathbb{R}^4) \longrightarrow END_{\mathbb{C}}(S_{\mathbb{C}})$ (IRREDUCIBLE)

2. $\Delta_{\mathbb{C}} : SPIN(4) \longrightarrow AUT_{\mathbb{C}}(S_{\mathbb{C}})$

THIS IS NOT IRREDUCIBLE AS ONE SEES BY WRITING

$$S_{\mathbb{C}} \cong S_{\mathbb{C}}^{+} \oplus S_{\mathbb{C}}^{-}$$

$$\begin{pmatrix} z^1 \\ z^2 \\ z^3 \\ z^4 \end{pmatrix} = \begin{pmatrix} z^1 \\ z^2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z^3 \\ z^4 \end{pmatrix}$$

CLIFFORD MULTIPLICATION BY ELEMENTS OF $Cl_0(\mathbb{R}^4)$ PRESERVES $S_{\mathbb{C}}^{+}$ AND $S_{\mathbb{C}}^{-}$, WHEREAS CLIFFORD MULTIPLICATION BY ELEMENTS OF $Cl_1(\mathbb{R}^4)$ REVERSES $S_{\mathbb{C}}^{+}$ AND $S_{\mathbb{C}}^{-}$. SPIN(4) $\subseteq Cl_0(\mathbb{R}^4) \Rightarrow$

$$\Delta_{\mathbb{C}} = \Delta_{\mathbb{C}}^{+} \oplus \Delta_{\mathbb{C}}^{-}$$

$$\Delta_{\mathbb{C}}^{\pm} : \mathrm{SPIN}(4) \longrightarrow SU(S_{\mathbb{C}}^{\pm})$$

$\Delta_{\mathbb{C}}^{+}$ AND $\Delta_{\mathbb{C}}^{-}$ ARE INEQUIVALENT, IRREDUCIBLE REPRESENTATIONS OF SPIN(4) ON $\mathbb{C}^2$.

3. $$\hat{\Delta}_{\mathbb{C}} : \mathrm{SPIN}^c(4) \longrightarrow \mathrm{AUT}_{\mathbb{C}}(S_{\mathbb{C}})$$

$$\hat{\Delta}_{\mathbb{C}} = \hat{\Delta}_{\mathbb{C}}^{+} \oplus \hat{\Delta}_{\mathbb{C}}^{-}$$

$$\hat{\Delta}_{\mathbb{C}}^{\pm} : \mathrm{SPIN}^c(4) \longrightarrow U(S_{\mathbb{C}}^{\pm})$$

A FEW MORE IMPORTANT ACTIONS :

4. $\mathbb{R}^4 \subseteq Cl_1(\mathbb{R}^4)$ ACTS BY CLIFFORD MULTIPLICATION ON $S_{\mathbb{C}}$, REVERSING $S_{\mathbb{C}}^+$ AND $S_{\mathbb{C}}^-$.

$$\mathbb{R}^4 \cdot (\,) : S_{\mathbb{C}}^{\pm} \longrightarrow S_{\mathbb{C}}^{\mp}$$

5. $SPIN^c(4)$ ACTS ON $\mathbb{C}$ VIA $\delta : SPIN^c(4) \to U(1)$ AND COMPLEX MULTIPLICATION.

$$\xi \cdot z = (e^{\theta i} u) \cdot z = \delta(\xi) z = e^{2\theta i} z$$

6. $SO(4)$ ACTS ON $Cl(\mathbb{R}^4)$:

SPIN(4) ACTS ON $Cl(\mathbb{R}^4)$ BY CONJUGATION

$$u \cdot p = u p u^{-1}$$

BUT $(-u) p (-u)^{-1} = u p u^{-1}$ SO THIS DESCENDS TO AN ACTION OF $SO(4) \cong SPIN(4)/\mathbb{Z}_2$ ON $Cl(\mathbb{R}^4)$.

7. COMPLEX-VALUED 2-FORMS $\Omega^2(\mathbb{R}^4)$ ON $\mathbb{R}^4$ ACT ON $S_{\mathbb{C}}$ VIA THE LINEAR ISOMORPHISM

$$\rho : \Omega^2(\mathbb{R}^4) \hookrightarrow Cl_0(\mathbb{R}^4) \otimes \mathbb{C}$$

$$\rho(\eta) = \rho\Big(\sum_{i<j} \eta_{ij} E^i \wedge E^j\Big) = \sum_{i<j} \eta_{ij} E_i E_j =$$

$$\begin{pmatrix} (\eta_{12}+\eta_{34}) I + (\eta_{13}-\eta_{24}) J + (\eta_{14}+\eta_{23}) K & 0 \\ 0 & (-\eta_{12}+\eta_{34}) I + (-\eta_{13}-\eta_{24}) J + (-\eta_{14}+\eta_{23}) K \end{pmatrix}$$

η REAL-VALUED $\Rightarrow$ $\rho(\eta)$ SKEW-HERMITIAN

η $\mathrm{Im}\,\mathbb{C}$-VALUED $\Rightarrow$ $\rho(\eta)$ HERMITIAN

$S^+_{\mathbb{C}}$ AND $S^-_{\mathbb{C}}$ INVARIANT UNDER ANY $\rho(\eta)$

$$\rho^{\pm}(\eta) = \rho(\eta)\,|\,S^{\pm}_{\mathbb{C}}$$

E.G., SUPPRESSING THE TWO ZERO ENTRIES

$$\rho^{+}(\eta) = (\eta_{12}+\eta_{34})I + (\eta_{13}+\eta_{42})J + (\eta_{14}+\eta_{23})K$$

$$\rho^{\pm} : \Omega^2(\mathbb{R}^4) \longrightarrow \mathrm{END}_{\mathbb{C}}(S^{\pm}_{\mathbb{C}})$$

$$\Omega^2(\mathbb{R}^4) \cong \Omega^2_+(\mathbb{R}^4) \oplus \Omega^2_-(\mathbb{R}^4)$$

$$\rho^{\pm}\,|\,\Omega^2_{\pm}(\mathbb{R}^4) \;:\; \Omega^2_{\pm}(\mathbb{R}^4) \twoheadrightarrow \mathrm{END}_0(S^{\pm}_{\mathbb{C}})$$

(COMPLEX LINEAR ISOMORPHISM)

INVERSES :

$$\sigma^{\pm} : \mathrm{END}_0(S^{\pm}_{\mathbb{C}}) \longrightarrow \Omega^2_{\pm}(\mathbb{R}^4)$$

<u>EXAMPLE</u> : LET $\psi = \begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix} \in S^+_{\mathbb{C}}$. GIVES AN ENDOMORPHISM OF $S^+_{\mathbb{C}}$:

$$\psi \otimes \psi^* = \begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}(\bar{\psi}^1 \;\; \bar{\psi}^2) = \begin{pmatrix} |\psi^1|^2 & \psi^1\bar{\psi}^2 \\ \bar{\psi}^1\psi^2 & |\psi^2|^2 \end{pmatrix}$$

TRACE FREE PART IS

$$(\psi \otimes \psi^*)_0 = \psi \otimes \psi^* - \tfrac{1}{2}\, tr(\psi \otimes \psi^*)\, \mathbb{1}$$

$$= \begin{pmatrix} \tfrac{1}{2}(|\psi^1|^2 - |\psi^2|^2) & \psi^1 \bar{\psi}^2 \\ \bar{\psi}^1 \psi^2 & \tfrac{1}{2}(|\psi^2|^2 - |\psi^1|^2) \end{pmatrix}$$

ONE VERIFIES THAT

$$\sigma^+((\psi \otimes \psi^*)_0) = -\tfrac{1}{4} \Big\{ (\psi^* I \psi)(E^1 \wedge E^2 + E^3 \wedge E^4) + (\psi^* J \psi)(E^1 \wedge E^3 + E^4 \wedge E^2) + (\psi^* K \psi)(E^1 \wedge E^4 + E^2 \wedge E^3) \Big\}$$

NOW GLOBALIZE THESE ALGEBRAIC CONSTRUCTIONS TO BUNDLES OVER

M = COMPACT, SIMPLY CONNECTED, ORIENTED SMOOTH 4-MANIFOLD + RESTRICTIONS ON $b_2^+(M)$

ANY CHOICE OF RIEMANNIAN METRIC g FOR M GIVES

$$SO(4) \hookrightarrow F_{SO}(M) \xrightarrow{\pi_{SO}} M$$

WE FIX THE LEVI-CIVITA CONNECTION ω_{LC} ON THIS FRAME BUNDLE.

<u>NOTE</u> : FOR MORE DETAILS ON THE CONSTRUCTIONS WHICH FOLLOW, SEE <u>ADDENDUM</u> <u>15</u> .

A <u>SPIN^c STRUCTURE</u> $\mathcal{L}$ FOR M CONSISTS OF A SPIN^c(4)-BUNDLE

$$\mathrm{SPIN}^c(4) \hookrightarrow S^c(M) \xrightarrow{\pi_{S^c}} M$$

OVER M AND A SMOOTH MAP $\Lambda : S^c(M) \longrightarrow F_{SO}(M)$ SATISFYING

$$\begin{tikzcd} S^c(M) \arrow[rr, "\Lambda"] \arrow[dr, "\pi_{S^c}"'] & & F_{SO}(M) \arrow[dl, "\pi_{SO}"] \\ & M & \end{tikzcd} \qquad \pi_{SO} \circ \Lambda = \pi_{S^c}$$

AND

$$\Lambda(p \cdot \xi) = \Lambda(p) \cdot \pi(\xi)$$

WHERE $\pi : \mathrm{SPIN}^c(4) \longrightarrow SO(4)$ IS MAP #2 ON P. 7.

<u>NOTE</u>: ANY COMPACT, ORIENTED 4-MANIFOLD ADMITS A SPIN^c STRUCTURE FOR ANY CHOICE OF g (HIRZEBRUCH-HOPF) AND, IN THE SIMPLY CONNECTED CASE, EQUIVALENCE CLASSES OF SPIN^c STRUCTURES ON M ARE IN ONE-TO-ONE CORRESPONDENCE WITH THE ELEMENTS OF $H^2(M;\mathbb{Z})$ WHOSE MOD 2 REDUCTION IS THE SECOND STIEFEL-WHITNEY CLASS $w_2(M)$ OF M.

GIVEN A SPIN^c STRUCTURE WE HAVE THE FOLLOWING ASSOCIATED BUNDLES:

$$\mathcal{S}(\mathcal{L}) = S^c(M) \times_{\hat{\Delta}_{\mathbb{C}}} S_{\mathbb{C}}$$ (<u>SPINOR</u> <u>BUNDLE</u>)

$$\mathcal{S}^{\pm}(\mathcal{L}) = S^c(M) \times_{\hat{\Delta}^{\pm}_{\mathbb{C}}} S^{\pm}_{\mathbb{C}}$$ (<u>POSITIVE</u> AND <u>NEGATIVE</u> <u>SPINOR</u> <u>BUNDLES</u>)

$$L(\mathcal{L}) = S^c(M) \times_{\delta} \mathbb{C}$$ (<u>DETERMINANT</u> <u>LINE</u> <u>BUNDLE</u>)

SOME OTHER RELEVANT BUNDLES :

$$U(1) \hookrightarrow L^{\circ}(\mathcal{L}) \xrightarrow{\pi_{L^{\circ}}} M$$ (ORIENTED, ORTHONORMAL FRAME BUNDLE FOR SOME FIBER METRIC ON $L(\mathcal{L})$)

$$Cl(M) = F_{SO}(M) \times_{SO(4)} Cl(4)$$ (<u>CLIFFORD</u> <u>BUNDLE</u>)

$$Cl(M) \otimes \mathbb{C} = F_{SO}(M) \times_{SO(4)} (Cl(4) \otimes \mathbb{C})$$ (<u>COMPLEXIFIED</u> <u>CLIFFORD</u> <u>BUNDLE</u>)

<u>NOTE</u> : CAN SHOW THAT $w_2(M) = c_1(L^{\circ}(\mathcal{L})) \bmod 2$ AND, CONVERSELY, GIVEN A PRINCIPAL $U(1)$-BUNDLE L° OVER M WITH $w_2(M) = c_1(L^{\circ}) \bmod 2$, THERE IS A SPIN^c STRUCTURE $\mathcal{L}$ FOR M WITH $L^{\circ}(\mathcal{L}) = L^{\circ}$.

GLOBAL VERSION OF $\sigma^+ : \mathrm{END}_0(S^+_{\mathbb{C}}) \longrightarrow \Omega^2_+(\mathbb{R}^4)$:

$$\sigma^+ : \Gamma(\mathrm{END}_0(\mathcal{S}^+(\mathcal{L}))) \longrightarrow \Omega^2_+(M)$$

IDENTIFIES SECTIONS OF THE TRACE FREE ENDOMORPHISM BUNDLE OF THE POSITIVE SPINOR BUNDLE WITH SELF-DUAL 2-FORMS ON M.

FIELD CONTENT OF SEIBERG - WITTEN THEORY :

GAUGE FIELD : CONNECTION A ON $U(1) \hookrightarrow L^{\circ}(\mathcal{L}) \rightarrow M$

POSITIVE SPINOR FIELD : SECTION $\psi \in \Gamma(S^{+}(\mathcal{L}))$

CURVATURE OF A IS THE $\mathfrak{u}(1)$-VALUED 2-FORM dA ON $L^{\circ}(\mathcal{L})$, BUT $U(1)$ IS ABELIAN SO THIS IS UNIQUELY DETERMINED BY A $\mathfrak{u}(1)$-VALUED 2-FORM F_A ON M (GAUGE POTENTIAL)

$$\psi \rightarrow (\psi \otimes \psi^{*})_{0} \in \Gamma(\mathrm{END}_{0}(S^{+}(\mathcal{L}))) \longrightarrow \sigma^{+}((\psi \otimes \psi^{*})_{0}) \in \Omega^{2}_{+}(M)$$

1ST SEIBERG-WITTEN EQUATION : $F_A^{+} = \sigma^{+}((\psi \otimes \psi^{*})_{0})$

FOR THE 2ND SEIBERG-WITTEN EQUATION WE INTRODUCE A DIRAC OPERATOR ON SPINOR FIELDS :

$SPIN^{c}(4)$ DOUBLE COVERS $SO(4) \times U(1)$ BY THE MAP $SPIN^{c} = (\pi, \delta)$ SO $S^{c}(M)$ DOUBLE COVERS THE FIBER PRODUCT $F_{SO}(M) \dot{\times} L^{\circ}(\mathcal{L})$. DENOTE THE MAP BY $SPIN^{c}$ ALSO.

$$\begin{tikzcd}
SPIN^{c}(4) \arrow[r, hook] & S^{c}(M) \arrow[r] \arrow[d, "SPIN^{c}"] & M \\
SO(4) \times U(1) \arrow[r, hook] & F_{SO}(M) \dot{\times} L^{\circ}(\mathcal{L}) \arrow[r] & M
\end{tikzcd}$$

THIS GIVES

$$\begin{array}{ccccc} SPIN^c(4) & \hookrightarrow & S^c(M) & \longrightarrow & M \\ & & \Big\downarrow {\scriptstyle Pr_F \circ SPIN^c} & & \\ SO(4) & \hookrightarrow & F_{SO}(M) & \longrightarrow & M \end{array} \qquad \begin{array}{ccccc} SPIN^c(4) & \hookrightarrow & S^c(M) & \longrightarrow & M \\ & & \Big\downarrow {\scriptstyle Pr_{L^0} \circ SPIN^c} & & \\ U(1) & \hookrightarrow & L^0(\mathcal{L}) & \longrightarrow & M \end{array}$$

FIXED CONNECTION : ω_{LC} SW CONNECTION : A

<u>SPIN</u>^{c} <u>CONNECTION</u>

$$\omega_A = (SPIN^c)^*(Pr_F^{\ *}\omega_{LC} + Pr_{L^0}^{\ *}A)$$

ω_A GIVES COVARIANT DERIVATIVES ON SECTIONS OF ASSOCIATED BUNDLES, E.G.,

$$\nabla_A : \Gamma(S(\mathcal{L})) \longrightarrow \Omega^1(M) \otimes \Gamma(S(\mathcal{L}))$$

$$\Psi \in \Gamma(S(\mathcal{L})) \rightarrow \nabla_A \Psi \in \Omega^1(M) \otimes \Gamma(S(\mathcal{L}))$$

$$\nabla_A \Psi(V) \in \Gamma(S(\mathcal{L}))$$

FOR EVERY VECTOR FIELD V ON M

LET $\{E_1, E_2, E_3, E_4\}$ BE A LOCAL ORIENTED, ORTHONORMAL FRAME FIELD ON M, I.E., A SECTION OF $F_{SO}(M)$.

EACH E_i CAN BE REGARDED EITHER AS A VECTOR FIELD ON M (SO $\nabla_A \Psi(E_i)$ MAKES SENSE) OR AS A SECTION OF THE CLIFFORD BUNDLE $Cl(M)$ WHICH THEREFORE ACTS BY CLIFFORD MULTIPLICATION ON SECTIONS OF $S(\mathcal{L})$.

$$\tilde{\not{\partial}}_A \Psi = \sum_{i=1}^{4} E_i \cdot \nabla_A \Psi(E_i)$$

$$S(\mathcal{L}) = S^+(\mathcal{L}) \oplus S^-(\mathcal{L})$$

CLIFFORD MULTIPLICATION BY E_i INTERCHANGES $S^{\pm}(\mathcal{L})$. RESTRICTING $\tilde{\not{D}}_A$ TO $S^{\pm}(\mathcal{L})$ THEREFORE GIVES

$$\not{D}_A : \Gamma(S^+(\mathcal{L})) \longrightarrow \Gamma(S^-(\mathcal{L}))$$

$$\not{D}_A^* : \Gamma(S^-(\mathcal{L})) \longrightarrow \Gamma(S^+(\mathcal{L}))$$

$\not{D}_A$ IS OUR DIRAC OPERATOR (AND $\not{D}_A^*$ IS ITS ADJOINT RELATIVE TO THE NATURAL INNER PRODUCTS ON SPACES OF SECTIONS).

2ND SEIBERG-WITTEN EQUATION : $\not{D}_A \psi = 0$

SEIBERG-WITTEN : GIVEN M, SELECT RIEMANNIAN METRIC g AND SPIN^c STRUCTURE $\mathcal{L}$ FOR THE CORRESPONDING $F_{SO}(M)$. SW CONFIGURATION SPACE IS

$$\mathcal{A}(\mathcal{L}) = \{ (A, \psi) : A \text{ IS A CONNECTION ON } U(1) \hookrightarrow L^0(\mathcal{L}) \to M \text{ AND } \psi \in \Gamma(S^+(\mathcal{L})) \text{ IS A POSITIVE SPINOR FIELD ON } M \}$$

$(A, \psi) \in \mathcal{A}(\mathcal{L})$ IS A SW MONOPOLE IF IT SATISFIES

(SW1) $\quad F_A^+ = \sigma^+((\psi \otimes \psi^*)_0)$, (CURVATURE EQUATION)

(SW2) $\quad \not{D}_A \psi = 0$ (DIRAC EQUATION)

NOTE : TO SEE WHAT THESE EQUATIONS LOOK LIKE ON $\mathbb{R}^4$, SOME EXPLICIT SOLUTIONS, AND SOME BOUNDEDNESS RESULTS, CONSULT ADDENDUM 16.

SEIBERG-WITTEN GAUGE GROUP $\mathcal{G}(\mathcal{L})$:

1. ALL AUTOMORPHISMS σ OF $S^c(M)$ THAT COVER THE IDENTITY ON $F_{SO}(M)$

$$Pr_F \circ SPIN^c \circ \sigma = Pr_F \circ SPIN^c$$

$$\begin{tikzcd} SPIN^c(4) \arrow[r, hook] & S^c(M) \arrow[r] \arrow[d] & M \\ SO(4) \arrow[r, hook] & F_{SO}(M) \arrow[r] & M \end{tikzcd}$$

2. $C^\infty(M, U(1))$

$$\gamma \in C^\infty(M, U(1)) \longleftrightarrow \sigma_\gamma : S^c(M) \to S^c(M)$$

$$\sigma_\gamma(p) = p \cdot \gamma(\pi_{S^c}(p))$$

ACTION OF $\mathcal{G}(\mathcal{L})$ ON $(A, \psi) \in \mathcal{A}(\mathcal{L})$:

CONNECTION A : σ_γ INDUCES AN AUTOMORPHISM OF $L^0(\mathcal{L})$

$$\begin{tikzcd} S^c(M) \arrow[r, "\sigma_\gamma"] \arrow[d] & S^c(M) \arrow[d] \\ L^0(\mathcal{L}) \arrow[r, dashed, "\sigma_\gamma'"'] & L^0(\mathcal{L}) \end{tikzcd}$$

$$A \cdot \gamma = (\sigma_\gamma')^* A$$

SPINOR FIELD ψ : IDENTIFY THE SECTION ψ OF $S^+(\mathcal{L})$ WITH AN EQUIVARIANT MAP $\psi : S^c(M) \to S^+_{\mathbb{C}}$

$$\psi \cdot \gamma = \sigma_\gamma^* \psi = \psi \circ \sigma_\gamma$$

$$(A, \psi) \cdot \gamma = (A \cdot \gamma, \psi \cdot \gamma) = ((\sigma_\gamma')^* A, \sigma_\gamma^* \psi)$$

THEOREM: THE ACTION OF $\mathcal{G}(\mathcal{L})$ ON THE SW CONFIGURATION SPACE $\mathcal{A}(\mathcal{L})$ CARRIES SOLUTIONS TO (SW) ONTO OTHER SOLUTIONS TO (SW), I.E., IF $(A, \psi) \in \mathcal{A}(\mathcal{L})$ SATISFIES

$$\begin{cases} F_A^+ = \sigma^+((\psi \otimes \psi^*)_0) \\ \not{\partial}_A \psi = 0 \end{cases}$$

AND $\gamma \in C^\infty(M, U(1))$, THEN $(A, \psi) \cdot \gamma = (A \cdot \gamma, \psi \cdot \gamma)$ SATISFIES

$$\begin{cases} F_{A\cdot\gamma}^{\;+} = \sigma^+(((\psi\cdot\gamma) \otimes (\psi\cdot\gamma)^*)_0) \\ \not{\partial}_{A\cdot\gamma}(\psi\cdot\gamma) = 0 \end{cases}$$

TWO MODULI SPACES:

$$\mathcal{B}(\mathcal{L}) = \mathcal{A}(\mathcal{L}) / \mathcal{G}(\mathcal{L})$$

$$\mathcal{M}(\mathcal{L}) = \left\{ (A, \psi) \in \mathcal{A}(\mathcal{L}) : F_A^+ = \sigma^+((\psi \otimes \psi^*)_0),\ \not{\partial}_A \psi = 0 \right\} / \mathcal{G}(\mathcal{L})$$

ASSUMING THAT EACH IS REPLACED BY AN "APPROPRIATE SOBOLEV COMPLETION" THE ANALYSIS IS VERY SIMILAR TO THAT OF THE DONALDSON MODULI SPACES.

WE WILL MENTION ONLY A FEW POINTS AT WHICH THERE ARE DIFFERENCES.

1. THE REDUCIBLE ELEMENTS OF $\mathcal{A}(\mathcal{L})$ ARE EASILY IDENTIFIED.

 LEMMA: AN ELEMENT (A, Ψ) OF $\mathcal{A}(\mathcal{L})$ IS LEFT FIXED BY SOME NON-IDENTITY ELEMENT γ OF $\mathcal{G}(\mathcal{L})$ IF AND ONLY IF $\Psi \equiv 0$ AND, IN THIS CASE, $\gamma : M \to U(1)$ MUST BE A CONSTANT MAP.

2. ASSOCIATED WITH ANY SOLUTION (A, Ψ) TO (SW) IS A FUNDAMENTAL ELLIPTIC COMPLEX $\mathcal{E}(A, \Psi)$ WITH FINITE DIMENSIONAL COHOMOLOGY GROUPS ADMITTING INTERPRETATIONS ANALOGOUS TO THOSE IN DONALDSON THEORY.

 $H^0(A, \Psi) = 0 \Leftrightarrow (A, \Psi)$ IRREDUCIBLE $\Leftrightarrow \Psi \not\equiv 0$

 $H^1(A, \Psi)$ = FORMAL TANGENT SPACE TO $\mathcal{M}(\mathcal{L})$ AT $[A, \Psi]$

 $H^2(A, \Psi)$ = OBSTRUCTION SPACE ($H^2(A, \Psi) = 0 \Leftrightarrow$ IMPLICIT FUNCTION THEOREM GIVES A LOCAL MANIFOLD STRUCTURE FOR THE SET OF SOLUTIONS TO (SW) NEAR (A, Ψ) OF DIMENSION $\dim H^1(A, \Psi)$)

IF $H^0(A, \Psi) = 0$ AND $H^2(A, \Psi) = 0$ THEN THE ATIYAH-SINGER THEOREM GIVES

$$\frac{1}{4}\left(c_1(L^0(\mathcal{L}))^2 - 2\chi(B) - 3\sigma(B) \right)$$

AS THE DIMENSION OF $\mathcal{M}(\mathcal{L})$ NEAR $[A, \Psi]$.

$H^0(A, \Psi) = 0$ AND $H^2(A, \Psi) = 0$ ARE THE GENERIC SITUATION, BUT NOW "GENERIC" MEANS SOMETHING DIFFERENT THAN IT DID IN DONALDSON THEORY.

(A) AS IN DONALDSON THEORY, IF $b_2^+(M) > 0$ THERE IS A DENSE G_δ SET $\mathrm{Gen}(\mathcal{R})$ OF RIEMANNIAN METRICS ON M SUCH THAT

FOR ANY $g \in \mathrm{Gen}(\mathcal{R})$ AND ANY CORRESPONDING SPIN^c STRUCTURE $\mathcal{L}$ ANY SOLUTION (A, Ψ) TO (SW) HAS

$$H^0(A, \Psi) = 0.$$

IF $b_2^+(M) > 1$ THIS IS TRUE FOR A GENERIC PATH OF RIEMANNIAN METRICS.

(B) THERE IS NO ANALOGOUS "GENERIC METRICS THEOREM" FOR $H^2(A,\Psi)$. IN THIS CASE ONE MUST PERTURB NOT THE METRIC, BUT THE EQUATIONS THEMSELVES.

FIX g AND $\mathcal{L}$. FOR ANY $\eta \in \Omega^2_+(M, \mathrm{Im}\,\mathbb{C})$, THE η-<u>PERTURBED SW EQUATIONS</u> ARE

$$F_A^+ = \sigma^+((\Psi \otimes \Psi^*)_0) + \eta$$

$$\not{\partial}_A \Psi = 0$$

EVERYTHING WE HAVE SAID ABOUT (SW) IS TRUE OF (PSW). ALSO

THERE IS A DENSE G_δ SET $\mathrm{Gen}(\Omega^2_+)$ IN $\Omega^2_+(M, \mathrm{Im}\,\mathbb{C})$ SUCH THAT FOR ANY $\eta \in \mathrm{Gen}(\Omega^2_+)$ EVERY SOLUTION (A, Ψ) TO (PSW) HAS

$$H^2(A, \Psi) = 0.$$

<u>THEOREM</u>: IF $b_2^+(M) > 0$, $g \in \mathrm{Gen}(\mathcal{R})$, $\mathcal{L}$ IS ANY CORRESPONDING SPIN^c STRUCTURE AND $\eta \in \mathrm{Gen}(\Omega^2_+)$, THEN THE MODULI SPACE $\mathcal{M}(\mathcal{L}, \eta)$ OF SOLUTIONS TO THE η-PERTURBED SEIBERG-WITTEN EQUATIONS IS EITHER EMPTY OR A SMOOTH MANIFOLD OF DIMENSION

$$\frac{1}{4}\left(c_1(L^\circ(\mathcal{L}))^2 - 2\chi(M) - 3\sigma(M)\right).$$

A CHOICE OF ORIENTATION FOR $H^2_+(M;\mathbb{R})$ CANONICALLY ORIENTS $\mathcal{M}(\mathcal{L}, \eta)$.

<u>NOTE</u>: IF $b_2^+(M) > 1$ THERE IS ALSO A "COBORDISM" RESULT FOR GENERIC PATHS OF METRICS AND PERTURBATIONS ANALOGOUS TO THAT IN DONALDSON THEORY (ONE CAN EFFECTIVELY "FIX" THE SPIN^c STRUCTURE ALONG A PATH OF METRICS BECAUSE THE FRAME BUNDLES ARE ALL NATURALLY ISOMORPHIC).

3. <u>THE SEIBERG-WITTEN MODULI SPACES ARE ALWAYS COMPACT</u> !

THIS IS THE MOST SIGNIFICANT DIFFERENCE BETWEEN DONALDSON AND SEIBERG-WITTEN THEORY.

ASD MODULI SPACES ARE NOT COMPACT BECAUSE THE ASD-EQUATIONS ARE CONFORMALLY INVARIANT IN DIMENSION 4.

BY CONTRAST, SEIBERG-WITTEN SOLUTIONS SATISFY <u>A PRIORI UNIFORM BOUNDS</u>, E.G., IF (A,ψ) SATISFIES (SW), THEN, FOR EVERY $x \in M$,

$$\|\psi(x)\|^2 \leq K(B) = \max\left\{-\tfrac{1}{2}\mathcal{K}(x_0) : x_0 \in M\right\}.$$

NOW FIX A GENERIC METRIC g, GENERIC PERTURBATION η AND ORIENTATION FOR $H^2_+(M;\mathbb{R})$. SUPPOSE THERE IS A CORRESPONDING SPIN^c STRUCTURE $\mathcal{L}$ FOR WHICH

$$c_1(L^0(\mathcal{L}))^2 = 2\chi(M) + 3\sigma(M).$$

THE MODULI SPACE IS EITHER EMPTY OR A FINITE SET OF ISOLATED POINTS, EACH EQUIPPED WITH A SIGN ± 1.

DEFINE THE <u>0-DIMENSIONAL</u> <u>SEIBERG-WITTEN</u> <u>INVARIANT</u>

$$SW_0(M,\mathcal{L})$$

OF M ASSOCIATED WITH $\mathcal{L}$ TO BE ZERO IN THE FIRST CASE AND THE SUM OF THE SIGNS IN THE SECOND.

> <u>NOTE</u> : WHEN $b_2^+(M) > 1$ A COBORDISM ARGUMENT SHOWS THAT $SW_0(M,\mathcal{L})$ IS INDEPENDENT OF g AND η AND IS, IN FACT, A DIFFERENTIAL TOPOLOGICAL INVARIANT OF M.

ANOTHER CONSEQUENCE OF THE <u>A PRIORI</u> BOUNDS :

> IF g AND η ARE FIXED, THEN THERE ARE ONLY FINITELY MANY (EQUIVALENCE CLASSES OF) SPIN^c STRUCTURES $\mathcal{L}$ ON B THAT SATISFY
>
> 1. $c_1(L^0(\mathcal{L}))^2 - 2\chi(M) - 3\sigma(M) \geq 0$, AND
> 2. $\mathcal{M}(\mathcal{L},\eta) \neq \phi$.

WHEN THE DIMENSION OF THE MODULI SPACE IS NOT ZERO, SEIBERG-WITTEN INVARIANTS ARE DEFINED BY INTEGRATING A CERTAIN COHOMOLOGY CLASS OVER IT:

FIX $p_0 \in M$. LET $\mathcal{G}_0(\mathcal{L})$ = SUBGROUP OF $\mathcal{G}(\mathcal{L})$ OF THOSE ELEMENTS THAT ACT TRIVIALLY ON FIBER OF $S^c(M)$ ABOVE p_0.

$$\mathcal{M}_0(\mathcal{L}, \eta) \; : \; \text{SW - SOLUTIONS MODULO } \mathcal{G}_0(\mathcal{L})$$

NATURAL PROJECTION : $\mathcal{M}_0(\mathcal{L}, \eta) \rightarrow \mathcal{M}(\mathcal{L}, \eta)$

U(1) - ACTION ON $\mathcal{M}_0(\mathcal{L}, \eta)$: $[A, \psi]_0 \cdot e^{\theta i} = [A, e^{-\theta i}\psi]_0$

PRINCIPAL U(1) - BUNDLE :

$$U(1) \hookrightarrow \mathcal{M}_0(\mathcal{L}, \eta) \longrightarrow \mathcal{M}(\mathcal{L}, \eta)$$

1ST CHERN CLASS :

$$c_1(\mathcal{M}_0(\mathcal{L}, \eta)) \in H^2(\mathcal{M}(\mathcal{L}, \eta); \mathbb{Z})$$

$$SW(M, \mathcal{L}) = \int_{\mathcal{M}(\mathcal{L}, \eta)} c_1(\mathcal{M}^0(\mathcal{L}, \eta))^{d_{\mathcal{L}}}$$

WHERE $2 d_{\mathcal{L}} = \frac{1}{4}\left(c_1(L^0(\mathcal{L}))^2 - 2\chi(M) - 3\sigma(M)\right)$.

EMPIRICAL EVIDENCE SUGGESTS THAT $SW(M, \mathcal{L}) \neq 0$ ONLY FOR THOSE $\mathcal{L}$ SATISFYING $c_1(L^0(\mathcal{L}))^2 - 2\chi(M) - 3\sigma(M) = 0$.

THE ELEMENTS $c_1(L^0(\mathcal{L})) \in H^2(M;\mathbb{Z})$ CORRESPONDING TO THOSE $\mathcal{L}$ FOR WHICH $c_1(L^0(\mathcal{L}))^2 - 2\chi(M) - 3\sigma(M) = 0$ ARE CALLED <u>SW</u> - <u>BASIC</u> <u>CLASSES</u>.

M IS SAID TO BE OF <u>SW</u> <u>SIMPLE</u> <u>TYPE</u> IF NONZERO SW-INVARIANTS ARISE ONLY FROM 0-DIMENSIONAL MODULI SPACES, I.E., IF

$$SW(M,\mathcal{L}) \neq 0 \implies c_1(L^0(\mathcal{L}))^2 - 2\chi(M) - 3\sigma(M) = 0.$$

$\boxed{\text{THE WITTEN CONJECTURE}}$

1. M IS OF D-SIMPLE TYPE $\iff$ M IS OF SW-SIMPLE TYPE
2. D-BASIC CLASSES COINCIDE WITH SW-BASIC CLASSES
3. $$\mathcal{D}_M(x) = \exp(Q_M(x,x)/2) \sum_{r=1}^{s} a_r \exp(K_r(x))$$
$$= \exp(Q_M(x,x)/2) \sum_{\mathcal{L}\in\Lambda} 2^{m(M)} SW_0(M,\mathcal{L}) \exp(c_1(L^0(\mathcal{L}))(x))$$

WHERE Λ IS THE SET OF ALL (EQUIVALENCE CLASSES OF) SPIN^c-STRUCTURES FOR WHICH $c_1(L^0(\mathcal{L}))^2 - 2\chi(M) - 3\sigma(M) = 0$ AND

$$m(M) = 2 + \frac{1}{4}(7\chi(M) + 11\sigma(M)).$$

<u>NOTE</u>: FOR A BRIEF TOUR OF HOW WITTEN ARRIVED AT THE CONJECTURE, SEE <u>ADDENDUM</u> <u>17</u> (BY MATILDE MARCOLLI).

ATTITUDES ONE MIGHT ADOPT TOWARD THIS CONJECTURE :

1. IT SHOULD BE RIGOROUSLY PROVED

   - PIDSTRIGATCH AND TYURIN

   - FEEHAN AND LENESS (PARTIAL RESULTS HAVE ALREADY PAID DIVIDENDS)

   - VAJIAC

2. RIGOROUSLY TRUE OR NOT, THE SW INVARIANTS PROVIDE A MUCH MORE TRACTABLE TOOL FOR THE STUDY OF 4-MANIFOLDS, SO IT MAKES GOOD, PRACTICAL SENSE TO ABANDON THE ASD EQUATIONS IN FAVOR OF THE SW EQUATIONS.

   - ESSENTIALLY EVERYONE ELSE

3. IF PHYSICS IS TRULY CAPABLE OF CASTING SUCH A PENETRATING LIGHT UPON MATHEMATICS AT THE DEEPEST LEVELS, MATHEMATICIANS WILL WANT TO TAKE HEED AND TURN THEIR ATTENTION ONCE AGAIN TO THEIR HISTORICAL ROOTS IN PHYSICS.

   - ATIYAH