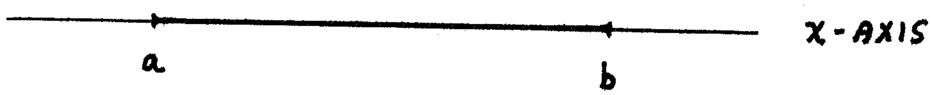
MASS AND AREA BETWEEN CURVES

WE WILL RETURN TO "TECHNIQUES OF INTEGRATION" (FINDING ANTIDERIVATIVES) A BIT LATER. FOR NOW WE WILL LOOK AT SOME SIMPLE

APPLICATIONS :

1. MASS

CONSIDER A THIN (1-DIMENSIONAL) METAL WIRE LYING ALONG THE X-AXIS FROM $x = a$ TO $x = b$.

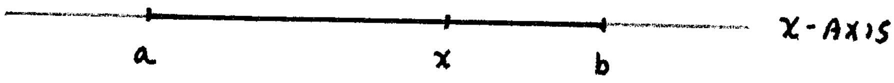


IF THE METAL IS HOMOGENEOUS (CONSTANT DENSITY, SAY, ρ_0 g/cm) THEN THE MASS M OF THE WIRE IS JUST

$$\begin{aligned} M &= (\#\ g/cm)(\#\ cm) \\ &= (\text{DENSITY})(\text{LENGTH}) \\ &= \rho_0 (b-a) \end{aligned}$$

SUPPOSE, HOWEVER, THAT THE METAL IS INHOMOGENEOUS (DENSITY VARIES FROM POINT-TO-POINT ALONG THE WIRE, SAY,

$$\rho = \rho(x) \ ).$$



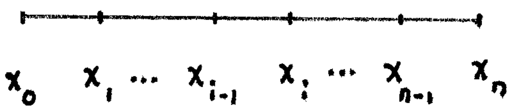
$\rho(x)$ = DENSITY AT LOCATION x
FOR EACH x IN $[a,b]$

HOW CAN ONE COMPUTE THE MASS M NOW ?

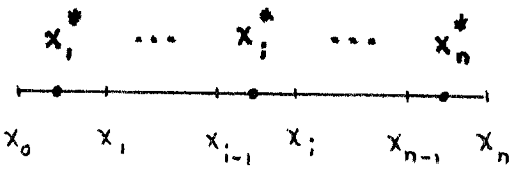
STANDARD OPERATING PROCEDURE : GENERATE A SEQUENCE OF BETTER AND BETTER APPROXIMATIONS TO M AND TAKE THEIR LIMIT.

IDEA : OVER A SMALL SEGMENT OF THE WIRE THE DENSITY IS NEARLY CONSTANT SO THE MASS OF THE SEGMENT IS APPROXIMATELY "DENSITY TIMES LENGTH". APPROXIMATION BECOMES BETTER AS LENGTH OF SEGMENT GOES TO ZERO.

1. CARVE THE WIRE UP INTO n SEGMENTS



2. SELECT A SAMPLE POINT IN EACH



EVALUATE THE DENSITY AT EACH SAMPLE POINT

$$\rho(x_1^*), \ldots, \rho(x_i^*), \ldots, \rho(x_n^*)$$

AND COMPUTE THE APPROXIMATE MASS OF EACH SEGMENT

$$\rho(x_1^*)\Delta x_1, \ldots, \rho(x_i^*)\Delta x_i, \ldots, \rho(x_n^*)\Delta x_n$$

3. COMPUTE THE APPROXIMATE MASS OF THE WIRE

$$\sum_{i=1}^{n} \rho(x_i^*)\Delta x_i$$

4. REPEAT # 1-3 OVER AND OVER WITH SMALLER AND SMALLER SEGMENTS AND TAKE THE LIMIT

$$M = \lim_{\Delta x_{max} \to 0} \sum_{i=1}^{n} \rho(x_i^*)\Delta x_i$$

$$= \int_a^b \rho(x)\,dx$$

<u>EXAMPLE</u> : A WIRE LIES ALONG THE X-AXIS FROM $x = 0$ TO $x = 1$ AND HAS A DENSITY AT EACH POINT THAT IS PROPORTIONAL TO ITS DISTANCE FROM THE LEFT-HAND ENDPOINT ($\rho(x) = kx$ FOR SOME CONSTANT k). THE MASS OF THE WIRE IS

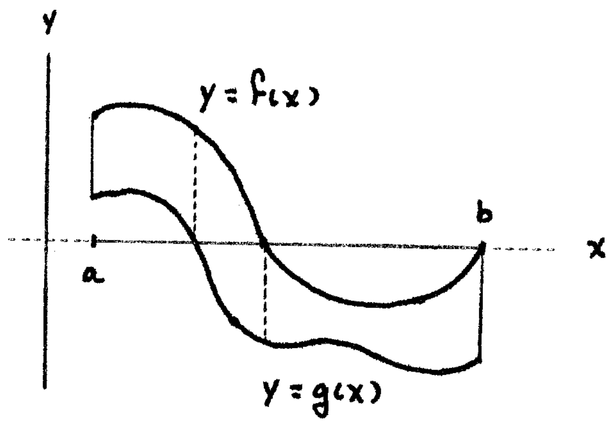
$$\int_0^1 \rho(x)\,dx = \int_0^1 kx\,dx = k\left[\tfrac{1}{2}x^2\Big|_0^1\right] = \frac{k}{2}$$

2. AREAS BETWEEN GRAPHS

SUPPOSE $f(x)$ AND $g(x)$ ARE CONTINUOUS ON $[a,b]$ AND

$$g(x) \le f(x)$$

FOR EACH x IN $[a,b]$.



AREA BETWEEN THE GRAPHS OF $f(x)$ AND $g(x)$ IS

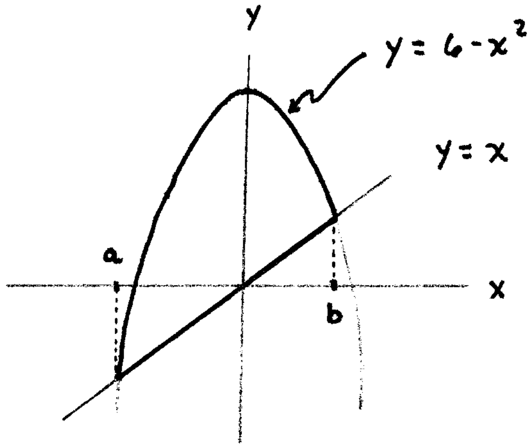
$$\int_a^b (f(x) - g(x))\,dx$$

TO SEE THIS, EXAMINE THE THREE PIECES (SHOWN ABOVE) SEPARATELY.

EXAMPLES :

1. COMPUTE THE AREA OF THE REGION BETWEEN THE GRAPHS OF $y = x$ AND $y = 6 - x^2$.

TO IDENTIFY THE TOP ($y = f(x)$), THE BOTTOM ($y = g(x)$) AND THE INTERVAL ($[a,b]$) WE NEED A SKETCH.



INTERSECTIONS :

$$6 - x^2 = x$$
$$x^2 + x - 6 = 0$$
$$(x+3)(x-2) = 0$$
$$x = -3, 2$$
$$[a,b] = [-3,2]$$

$$\text{AREA} = \int_{-3}^{2} \left((6-x^2) - x \right) dx = \int_{-3}^{2} (6 - x^2 - x) dx$$

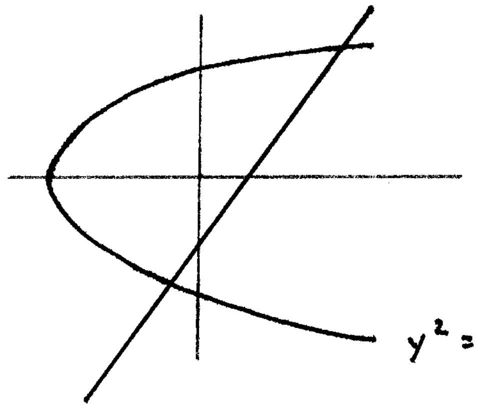
$$= 6x \Big|_{-3}^{2} - \frac{1}{3}x^3 \Big|_{-3}^{2} - \frac{1}{2}x^2 \Big|_{-3}^{2}$$

$$= 6(2-(-3)) - \frac{1}{3}(8-(-27)) - \frac{1}{2}(4-9)$$

$$= \frac{125}{6}$$

2. COMPUTE THE AREA OF THE REGION BETWEEN THE GRAPHS OF $y = x - 1$ AND $y^2 = 2x + 6$.

SKETCH : $y^2 = 2x + 6$ $\left(x = \frac{1}{2} y^2 - 3 \right)$ IS A PARABOLA AROUND THE X-AXIS.



THE "BOTTOM" (IN THE y-DIRECTION) CHANGES. COULD SPLIT THE AREA INTO TWO PIECES AND EVALUATE EACH AS WE DID IN EXAMPLE # 1.

MORE CONVENIENT TO REVERSE THE ROLES OF THE VARIABLES.

BOTTOM (IN X-DIRECTION) : $x = \frac{1}{2} y^2 - 3$

TOP (IN X-DIRECTION) : $x = y + 1$

INTERSECTIONS :

$$\frac{1}{2} y^2 - 3 = y + 1$$

$$y^2 - 2y - 8 = 0$$

$$(y + 2)(y - 4) = 0$$

$$y = -2, 4$$

$$\text{AREA} = \int_{-2}^{4} ((y+1) - (\tfrac{1}{2}y^2 - 3))\,dy$$

$$= \int_{-2}^{4} (-\tfrac{1}{2}y^2 + y + 4)\,dy$$

$$= -\tfrac{1}{6}y^3\Big|_{-2}^{4} + \tfrac{1}{2}y^2\Big|_{-2}^{4} + 4y\Big|_{-2}^{4}$$

$$= 18$$