CALCULUS AND GEOMETRY PROBLEMS

EXAMPLES:

1. LET E BE A NONDEGENERATE ELLIPSE IN THE PLANE. LET Δ DENOTE THE LARGEST OF THE AREAS OF TRIANGLES INSCRIBED IN E. HOW MANY INSCRIBED TRIANGLES HAVE THIS MAXIMAL AREA Δ (ONE, TWO, --- INFINITELY MANY)?

CHOOSE COORDINATES IN THE PLANE SO THAT THE EQUATION OF E IS

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

WITH $a, b > 0$. THE AREA OF THE ELLIPSE IS THEN πab.

NOTICE THAT THE LINEAR TRANSFORMATION T GIVEN BY

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{b}{a}} & 0 \\ 0 & \sqrt{\frac{a}{b}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

HAS DETERMINANT 1 (SO T AND T^{-1} BOTH PRESERVE AREA) AND CARRIES E ONTO

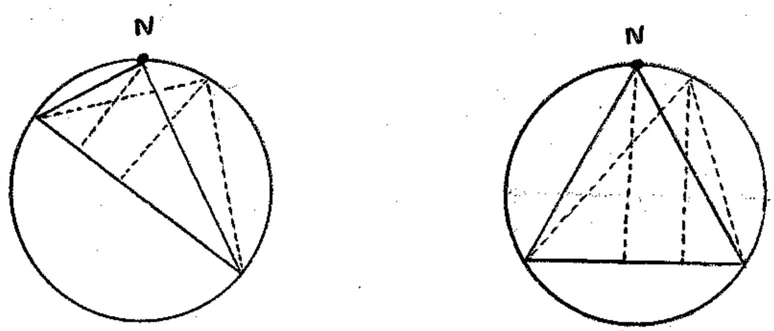
$$\begin{aligned}(x')^2 + (y')^2 &= \left(\sqrt{\tfrac{b}{a}}\,x\right)^2 + \left(\sqrt{\tfrac{a}{b}}\,y\right)^2 \\ &= \frac{b}{a}x^2 + \frac{a}{b}y^2 \\ &= (ab)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) \\ &= ab\end{aligned}$$

(A CIRCLE WITH THE SAME AREA AS E).

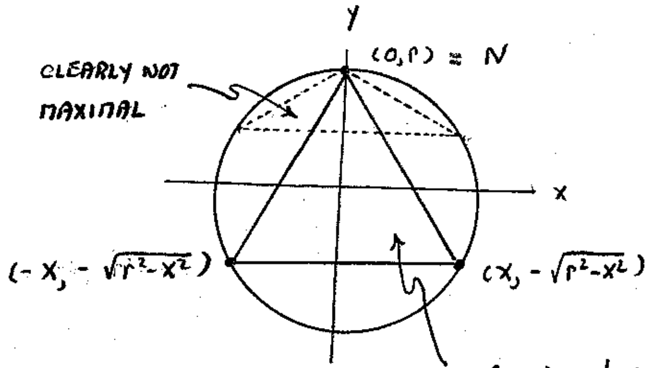
ANY TRIANGLE INSCRIBED IN E IS CARRIED BY T ONTO A TRIANGLE INSCRIBED IN THE CIRCLE WITH THE SAME AREA. CONVERSELY, ANY TRIANGLE INSCRIBED IN THE CIRCLE IS CARRIED BY T^{-1} ONTO A TRIANGLE INSCRIBED IN E WITH THE SAME AREA. MOREOVER, SINCE T AND T^{-1} ARE ISOMORPHISMS OF $\mathbb{R}^2$ ONTO $\mathbb{R}^2$ THIS CORRESPONDENCE BETWEEN INSCRIBED TRIANGLES IS A BIJECTION.

WE SHOW THAT ANY CIRCLE HAS INFINITELY MANY INSCRIBED TRIANGLES OF MAXIMAL AREA SO THE SAME MUST BE TRUE OF E.

CONSIDER A CIRCLE $x^2+y^2=r^2$ AND A FIXED POINT ON IT (SAY, THE NORTH POLE $N=(0,r)$). WE CONSIDER INSCRIBED TRIANGLES WITH ONE VERTEX AT N. SUPPOSE THERE IS SUCH A TRIANGLE OF MAXIMAL AREA. THEN THE SIDE OPPOSITE N MUST BE HORIZONTAL SINCE OTHERWISE WE COULD PRODUCE AN INSCRIBED TRIANGLE OF LARGER ALTITUDE WITH THE SAME BASE (AND THEREFORE LARGER AREA).



AMONG THE INSCRIBED TRIANGLES WITH N AS ONE VERTEX AND OPPOSITE SIDE HORIZONTAL THERE IS, IN FACT, ONE OF MAXIMAL AREA THAT CAN BE FOUND BY THE METHODS OF ELEMENTARY CALCULUS:



$$A(x) = \frac{1}{2}(2x)\left(r + \sqrt{r^2 - x^2}\right)$$

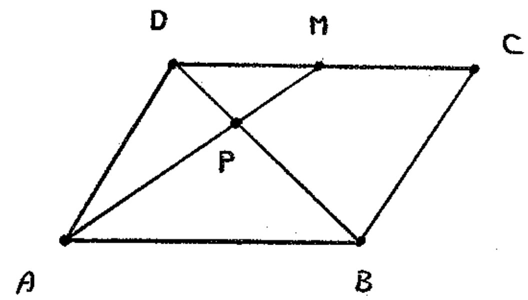
$$0 \le x \le r$$

(CONTINUOUS FUNCTION ON A CLOSED, BOUNDED INTERVAL SO THERE IS A MAXIMUM)

THIS THEN HAS THE MAXIMAL AREA OF ANY INSCRIBED TRIANGLE WITH ONE VERTEX AT N. BUT ROTATION ALSO PRESERVES AREAS SO THERE IS A TRIANGLE OF MAXIMAL AREA WITH A VERTEX AT ANY POINT OF THE CIRCLE. WE THEREFORE HAVE INFINITELY MANY INSCRIBED TRIANGLES OF MAXIMAL AREA.

2. PROVE THAT THE LINE JOINING ONE VERTEX OF A PARALLELOGRAM TO THE MIDPOINT OF AN OPPOSITE SIDE TRISECTS A DIAGONAL OF THE PARALLELOGRAM.

WE WILL TURN THIS INTO A PROBLEM IN VECTOR ALGEBRA. HERE IS THE PICTURE :



M IS THE MIDPOINT OF DC

WE SHOW THAT $\vec{BP} = \frac{2}{3}\vec{BD}$.

NOTE THAT $\vec{AB} = \vec{DC}$ AND $\vec{AD} = \vec{BC}$. MOREOVER, FOR SOME CONSTANTS α AND β,

$$\vec{AP} = \alpha \vec{AM}$$
$$\vec{AP} = \vec{AB} + \beta \vec{BD}$$

SO

$$\alpha \vec{AM} = \vec{AB} + \beta \vec{BD}$$
$$\alpha(\vec{AD} + \tfrac{1}{2}\vec{DC}) = \vec{AB} + \beta(\vec{AD} - \vec{AB})$$
$$\alpha(\vec{AD} + \tfrac{1}{2}\vec{AB}) = \vec{AB} + \beta(\vec{AD} - \vec{AB})$$
$$(\tfrac{1}{2}\alpha - 1 + \beta)\vec{AB} + (\alpha - \beta)\vec{AD} = \vec{0}$$

BUT $\vec{AB}$ AND $\vec{AD}$ ARE LINEARLY INDEPENDENT SO THIS IMPLIES

$$\begin{cases} \tfrac{1}{2}\alpha - 1 + \beta = 0 \\ \alpha - \beta = 0 \end{cases}$$

SO $\alpha = \beta = \frac{2}{3}$. THUS,

$$\vec{AP} = \vec{AB} + \tfrac{2}{3}\vec{BD}$$
$$\vec{AP} - \vec{AB} = \tfrac{2}{3}\vec{BD}$$
$$\vec{BP} = \tfrac{2}{3}\vec{BD}.$$

3. SHOW THAT THE EQUATION

$$X^2 = X \sin X + \cos X$$

HAS EXACTLY TWO REAL ROOTS.

NOTE THAT

$$f(x) = x^2 - x \sin x - \cos x$$

IS CONTINUOUS ON $\mathbb{R}$ AND SATISFIES

$$f(-\tfrac{\pi}{2}) = \frac{\pi^2}{4} + \frac{\pi}{2} > 0$$

$$f(0) = -1 < 0$$

$$f(\tfrac{\pi}{2}) = \frac{\pi^2}{4} - \frac{\pi}{2} > 0$$

SO, BY THE INTERMEDIATE VALUE THEOREM, $f(x)$ HAS ZEROS IN $(-\frac{\pi}{2}, 0)$ AND IN $(0, \frac{\pi}{2})$. THE EQUATION THEREFORE HAS AT LEAST TWO REAL ROOTS, SAY, x_1 AND x_2.

BY ROLLE'S THEOREM, $f'(x)$ MUST VANISH AT SOME POINT a IN (x_1, x_2). SUPPOSE $f(x)$ HAD A THIRD ZERO x_3. THEN $f'(x)$ WOULD HAVE TO VANISH ALSO ON SOME INTERVAL NOT CONTAINING a, E.G., (x_3, x_1), (x_2, x_3), (x_1, x_3), OR (x_3, x_2). BUT

$$f'(x) = 2x - \sin x - x \cos x + \sin x$$

$$= x(2 - \cos x)$$

VANISHES ONLY AT $x = 0$ SO THIS IS IMPOSSIBLE. THUS, THE EQUATION HAS EXACTLY TWO REAL ROOTS.

4. FIND ALL CONTINUOUSLY DIFFERENTIABLE FUNCTIONS $f : \mathbb{R} \to \mathbb{R}$ SATISFYING

$$(f(x))^2 = \int_0^x ((f(t))^2 + (f'(t))^2)\,dt + 2011$$

(PROVE THAT YOU HAVE FOUND THEM ALL).

NOTE THAT $(f(0))^2 = 0 + 2011 \Rightarrow$

$$f(0) = \pm\sqrt{2011}$$

MOREOVER, DIFFERENTIATING BOTH SIDES OF $(f(x))^2 = \int_0^x ((f(t))^2 + (f'(t))^2)\,dt + 2011$ GIVES

$$2f(x)f'(x) = (f(x))^2 + (f'(x))^2$$

FOR ALL X. REWRITE THIS AS

$$(f(x))^2 - 2f(x)f'(x) + (f'(x))^2 = 0$$

$$(f(x) - f'(x))^2 = 0$$

$$f'(x) = f(x)$$

THUS, THE FUNCTIONS WE SEEK ARE THE <u>UNIQUE</u> SOLUTIONS TO THE INITIAL VALUE PROBLEMS

$$\begin{cases} f'(x) = f(x) \\ f(0) = \sqrt{2011} \end{cases} \quad \text{AND} \quad \begin{cases} f'(x) = f(x) \\ f(0) = -\sqrt{2011} \end{cases}$$

THESE ARE EASILY SOLVED TO GIVE

$$f(x) = \pm\sqrt{2011}\, e^x .$$