GENERATING FUNCTIONS

THE IDEA : STUDY A SEQUENCE

$$a_0, a_1, a_2, \ldots, a_n, \ldots$$

BY LOOKING INSTEAD AT THE FUNCTION $A(x)$ WHOSE POWER SERIES EXPANSION

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

HAS THIS SEQUENCE AS ITS SEQUENCE OF COEFFICIENTS (THE GENERATING FUNCTION FOR THE SEQUENCE).

E.G., WE WILL SHOW THAT THE GENERATING FUNCTION $F(x)$ FOR THE FIBONACCI SEQUENCE, DEFINED BY

$$F_0 = 0$$
$$F_1 = 1$$
$$F_{n+1} = F_n + F_{n-1}, \quad n \geq 1,$$

IS

$$F(x) = \frac{x}{1-x-x^2}$$

(SEE EXAMPLE 2).

EXAMPLES :

1. DEFINE A SEQUENCE $a_0, a_1, a_2, \ldots$ BY

$$a_0 = 1$$
$$a_{n+1} = 2a_n + n, \quad n \geq 0.$$

FIND A CLOSED FORM EXPRESSION FOR a_n, $n \geq 0$.

COMPUTING THE FIRST FEW TERMS GIVES

$$1, 2, 5, 12, 27, 58, 121, \ldots$$

SO THE PATTERN IS NOT IMMEDIATELY APPARENT (IF IT HAD BEEN, WE COULD "GUESS" a_n AND TRY TO PROVE THAT OUR GUESS IS CORRECT, SAY, BY INDUCTION).

INSTEAD WE WILL TRY TO FIND A FUNCTION $A(x)$ WHOSE POWER SERIES EXPANSION IS $\sum_{n=0}^{\infty} a_n x^n$, WHERE THE a_n SATISFY THE GIVEN RECURRENCE FORMULA.

FOR $n \geq 0$,

$$a_{n+1} = 2a_n + n \implies a_{n+1} x^n = 2a_n x^n + n x^n$$

NOW SUM OVER $n = 0, 1, 2, \ldots$

$$\sum_{n=0}^{\infty} a_{n+1} x^n = 2 \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} n x^n$$

NOTE: WE DO NOT YET KNOW THE a_n SO WE CANNOT ADDRESS ISSUES OF CONVERGENCE AT THIS POINT. THIS CAN ALWAYS BE DONE AFTER WE DETERMINE $A(x)$, BUT IN MANY INSTANCES THE DETAILS ARE NOT SIGNIFICANT; CONVERGENCE ON ANY INTERVAL WILL JUSTIFY THE MANIPULATIONS WE PERFORM WITH THE COEFFICIENTS AND ONLY THESE COEFFICIENTS ARE OF INTEREST.

NOW WE EXAMINE THE LEFT- AND RIGHT-HAND SIDES SEPARATELY.

$$\sum_{n=0}^{\infty} a_{n+1} x^n = a_1 + a_2 x + a_3 x^2 + \cdots = \frac{a_1 x + a_2 x^2 + a_3 x^3 + \cdots}{x}$$

$$= \frac{A(x) - a_0}{x}$$

$$= \frac{A(x) - 1}{x}$$

$$2 \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} n x^n = 2A(x) + \sum_{n=0}^{\infty} n x^n$$

FOR $\sum_{n=0}^{\infty} n x^n$: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ FOR $|x| < 1$

DIFFERENTIATE :

$$\sum_{n=0}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2} \Rightarrow$$

$$\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

THUS,

$$\frac{A(x) - 1}{x} = 2A(x) + \frac{x}{(1-x)^2}$$

SOLVING FOR $A(x)$ GIVES

$$A(x) = \frac{2x^2 - 2x + 1}{(1-x)^2 (1-2x)}$$

AND THIS IS THE GENERATING FUNCTION FOR OUR SEQUENCE.

NOW WE TURN MATTERS AROUND AND EXPLICITLY DETERMINE THE POWER SERIES EXPANSION OF $A(x)$. PARTIAL FRACTIONS GIVES

$$A(x) = \frac{-1}{(1-x)^2} + \frac{2}{1-2x}$$

$$= -\frac{d}{dx}\left(\frac{1}{1-x}\right) + 2\left(\frac{1}{1-2x}\right)$$

$$A(x) = -\frac{d}{dx}\sum_{n=0}^{\infty} x^n + 2\sum_{n=0}^{\infty} (2x)^n$$

$$= \sum_{n=1}^{\infty} -n x^{n-1} + \sum_{n=0}^{\infty} 2^{n+1} x^n$$

$$= \sum_{n=0}^{\infty} -(n+1) x^n + \sum_{n=0}^{\infty} 2^{n+1} x^n$$

$$= \sum_{n=0}^{\infty} (2^{n+1} - n - 1) x^n$$

CONCLUSION :

$$a_n = 2^{n+1} - n - 1 \ , \quad n \geq 0$$

NOTE : A NICE EXERCISE AT THIS POINT WOULD BE TO PRETEND THAT YOU "GUESSED" THIS AND PROVE IT BY INDUCTION FROM THE RECURRENCE FORMULA.

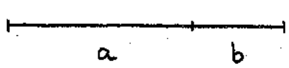
2. THE FIBONACCI NUMBERS F_n , $n \geq 0$, ARE DEFINED BY

$$F_0 = 0$$
$$F_1 = 1$$
$$F_{n+1} = F_n + F_{n-1} \ , \quad n \geq 1$$

SHOW THAT

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} \ , \quad n \geq 0.$$

NOTE : YOU PROBABLY RECOGNIZE $\frac{1+\sqrt{5}}{2}$ AS THE GOLDEN RATIO :



$$\frac{a+b}{a} = \frac{a}{b} \Rightarrow \frac{a}{b} = \frac{1+\sqrt{5}}{2}.$$

WE FIND THE GENERATING FUNCTION

$$F(x) = \sum_{n=0}^{\infty} F_n x^n = F_0 + F_1 x + F_2 x^2 + \cdots = F_1 x + F_2 x^2 + \cdots$$

FOR $n \geq 1$,

$$F_{n+1} = F_n + F_{n-1} \Rightarrow F_{n+1} x^n = F_n x^n + F_{n-1} x^n \Rightarrow$$

$$\sum_{n=1}^{\infty} F_{n+1} x^n = \sum_{n=1}^{\infty} F_n x^n + \sum_{n=1}^{\infty} F_{n-1} x^n \Rightarrow$$

$$F_2 x + F_3 x^2 + \cdots = F(x) + F_0 x + F_1 x^2 + F_2 x^3 + \cdots$$

$$\frac{F_2 x^2 + F_3 x^3 + \cdots}{x} = F(x) + 0 + x(F_1 x + F_2 x^2 + \cdots)$$

$$\frac{F(x) - F_1 x}{x} = F(x) + x F(x)$$

$$\frac{F(x) - x}{x} = F(x) + x F(x)$$

SOLVING FOR $F(x)$ GIVES

$$F(x) = \frac{x}{1 - x - x^2}$$

NOW WE FIND THE POWER SERIES EXPANSION FOR $F(x)$ DIRECTLY.

TO APPLY PARTIAL FRACTIONS FIRST FACTOR $1-x-x^2$ BY SOLVING

$$1 - x - x^2 = 0$$

QUADRATIC FORMULA GIVES

$$x = -\frac{1 \pm \sqrt{5}}{2}$$

THUS,

$$1 - x - x^2 = \left(1 - \left(\tfrac{1+\sqrt{5}}{2}\right)x\right)\left(1 - \left(\tfrac{1-\sqrt{5}}{2}\right)x\right)$$

PARTIAL FRACTIONS GIVES

$$F(x) = \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\left(\frac{1+\sqrt{5}}{2}\right)x} - \frac{1}{1-\left(\frac{1-\sqrt{5}}{2}\right)x}\right)$$

$$= \frac{1}{\sqrt{5}}\left(\sum_{n=0}^{\infty}\left(\frac{1+\sqrt{5}}{2}\right)^n x^n - \sum_{n=0}^{\infty}\left(\frac{1-\sqrt{5}}{2}\right)^n x^n\right)$$

$$= \sum_{n=0}^{\infty}\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)x^n$$

SO

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

FOR $n \geq 0$, AS REQUIRED.