INEQUALITIES

WE WILL DESCRIBE JUST A FEW OF THE MORE ELEMENTARY TECHNIQUES FOR PROVING INEQUALITIES BASED ON SIMPLE ALGEBRAIC MANIPULATION, SOME STANDARD RESULTS FROM CALCULUS, COMBINATORIAL IDENTITIES, INFINITE SERIES, AND A FEW "WELL-KNOWN" INEQUALITIES THAT ARE WORTH REMEMBERING.

NOTE : EACH OF THE FOLLOWING CAN BE VASTLY GENERALIZED, BUT WE WILL MAKE DO WITH THE SIMPLEST VERSIONS.

TRIANGLE INEQUALITY : $x_1, \dots, x_n \in \mathbb{R} \Rightarrow$

$$|x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n|$$

CAUCHY - SCHWARZ INEQUALITY : $x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n) \in \mathbb{R}^n \Rightarrow$

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

I.E.,

$$|x_1 y_1 + \cdots + x_n y_n| \leq \sqrt{x_1^2 + \cdots + x_n^2}\ \sqrt{y_1^2 + \cdots + y_n^2}$$

WITH EQUALITY HOLDING IFF x AND y ARE LINEARLY DEPENDENT.

ARITHMETIC - GEOMETRIC MEAN INEQUALITY : $x_1, \dots, x_n > 0 \Rightarrow$

$$\sqrt[n]{x_1 \cdots x_n} \leq \frac{x_1 + \cdots + x_n}{n}$$

EXAMPLES :

1. LET X, Y AND Z BE POSITIVE REAL NUMBERS. PROVE THAT

$$x^2+y^2+z^2 \geq xy+yz+xz$$

$$0 \leq (x-y)^2+(y-z)^2+(x-z)^2 = x^2-2xy+y^2+y^2-2yz+z^2+x^2-2xz+z^2$$
$$= 2x^2+2y^2+2z^2-2xy-2yz-2xz$$

$$\Rightarrow \quad 2x^2+2y^2+2z^2 \geq 2xy+2yz+2xz$$

$$\Rightarrow \quad x^2+y^2+z^2 \geq xy+yz+xz$$

2. LET a, b AND c BE POSITIVE REAL NUMBERS WITH $(1+a)(1+b)(1+c) = 8$, PROVE THAT $abc \leq 1$.

$(1+a)(1+b)(1+c) = 8 \Rightarrow (1+a+b+ab)(1+c) = 8 \Rightarrow$

$1+a+b+ab+c+ac+bc+abc = 8 \Rightarrow$

$$1+(a+b+c)+(ab+bc+ac)+abc = 8$$

ARITHMETIC-GEOMETRIC MEAN INEQUALITY $\Rightarrow$

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc} \Rightarrow a+b+c \geq 3\sqrt[3]{abc}$$

AND

$$\frac{ab+bc+ac}{3} \geq \sqrt[3]{(ab)(bc)(ac)} \Rightarrow ab+bc+ac \geq 3\left(\sqrt[3]{abc}\right)^2$$

SO

$$8 = 1 + (a+b+c) + (ab+bc+ac) + abc$$

$$\geq 1 + 3\sqrt[3]{abc} + 3(\sqrt[3]{abc})^2 + (\sqrt[3]{abc})^3 = (1+\sqrt[3]{abc})^3$$

THUS,

$$2 \geq 1 + \sqrt[3]{abc}$$

AND SO

$$\sqrt[3]{abc} \leq 1,$$

I.E.,

$$abc \leq 1$$

3. LET m AND n BE POSITIVE INTEGERS. PROVE THAT

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n} .$$

THE BINOMIAL THEOREM GIVES

$$(m+n)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} m^k n^{(m+n)-k}$$

$$= \binom{m+n}{m} m^m n^n + \text{POSITIVE TERMS}$$

$$> \binom{m+n}{m} m^m n^n = \frac{(m+n)!}{m!\, n!} m^m n^n$$

SO

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n} .$$

4. PROVE THAT, FOR ANY $0 \le p \le 1$ AND ANY REAL NUMBERS a AND b,

$$|a+b|^p \le |a|^p + |b|^p.$$

NOTE THAT THE RESULT IS TRIVIAL IF $p=0$ $(1 \le 2)$ AND, IF $p=1$, IT IS JUST THE TRIANGLE INEQUALITY SO WE CAN ASSUME

$$0 < p < 1.$$

THE INEQUALITY IS OBVIOUS IF EITHER a OR b IS ZERO. IT IS ALSO CLEAR IF a AND b HAVE OPPOSITE SIGNS SINCE THEN EITHER $|a+b| \le |a|$ OR $|a+b| \le |b|$, E.G., IF $a>0$ AND $b<0$, THEN $|a+b| = \begin{cases} a+b, & a+b \geq 0 \\ -a-b, & a+b \le 0 \end{cases} = \begin{cases} |a|-|b|, & a+b \geq 0 \\ -|a|+|b|, & a+b \le 0 \end{cases}$. FINALLY, IT IS ENOUGH TO PROVE THE RESULT WHEN a AND b ARE BOTH POSITIVE SINCE $|a+b|^p = |-a-b|^p$, $|a|^p = |-a|^p$ AND $|b|^p = |-b|^p$. THUS, WE CAN ASSUME THAT

$$a, b > 0.$$

WE ARE TO PROVE THAT

$$(a+b)^p \le a^p + b^b,$$

WHICH WE WRITE AS

$$(1+\frac{b}{a})^p \le 1 + (\frac{b}{a})^p.$$

IN FACT, WE WILL PROVE MORE GENERALLY THAT

$$0<p<1 \text{ AND } X>0 \implies (1+X)^p < 1+X^p.$$

FOR THIS IT WILL SUFFICE TO SHOW THAT, ON $X>0$, THE FUNCTION

$$f(x) = 1+x^p - (1+x)^p$$

IS POSITIVE. NOTE THAT $f(x)$ IS CONTINUOUS ON $X \geq 0$ AND

$$f(0) = 0$$

MOREOVER, ON $x > 0$,

$$f'(x) = px^{p-1} - p(1+x)^{p-1} = p\left(x^{p-1} - (1+x)^{p-1}\right),$$

$$= p\left[\frac{1}{x^{1-p}} - \frac{1}{(1+x)^{1-p}}\right] \qquad (0 < 1-p < 1)$$

$$> 0$$

SO $f(x)$ IS INCREASING ON $x > 0$. IT FOLLOWS FROM THE MEAN VALUE THEOREM THAT $f(x) > 0$ FOR $x > 0$, AS REQUIRED.

5. SHOW THAT FOR EVERY REAL NUMBER x AND EVERY INTEGER $n \geq 0$,

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{2n}}{(2n)!} > 0.$$

THE INEQUALITY IS OBVIOUS IF $x \geq 0$. FOR $x < 0$ WE NOTE THAT THE SUM ON THE LEFT-HAND SIDE IS THE PARTIAL SUM S_{2n} OF THE SERIES $\sum_{k=0}^{\infty} \frac{1}{k!} x^k$, WHICH CONVERGES TO e^x AND IS AN ALTERNATING SERIES. BUT THE SUM OF AN ALTERNATING SERIES $\sum_{k=0}^{\infty} (-1)^k a_k$ WITH a_k DECREASING TO 0 IS BETWEEN ANY TWO CONSECUTIVE PARTIAL SUMS AND, SINCE $\frac{x^{2n}}{(2n)!} > 0$ AND $S_{2n-1} < S_{2n}$, WE CONCLUDE THAT

$$S_{2n-1} < e^x < S_{2n}.$$

BUT $e^x > 0$ SO $S_{2n} > 0$.