MATHEMATICAL INDUCTION

LET $P(n)$ BE A STATEMENT ABOUT THE POSITIVE INTEGER n (E.G., "n CAN BE WRITTEN AS A PRODUCT OF PRIMES", OR "$1+2+\cdots+n = \frac{n(n+1)}{2}$")

TO PROVE THAT $P(n)$ IS TRUE FOR ALL $n \geq n_0$ IT SUFFICES TO SHOW THAT

(a) $P(n_0)$ IS TRUE, AND

(b) FOR $k \geq n_0$, IF $P(k)$ IS TRUE, THEN $P(k+1)$ MUST ALSO BE TRUE.

ALTERNATIVE TO (b):

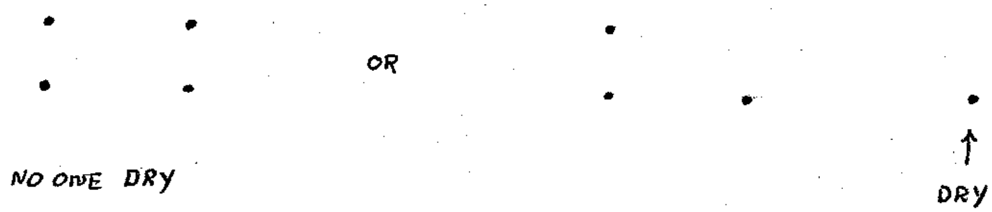
(b') IF $P(n)$ IS TRUE FOR ALL $n_0 \leq n < k$, THEN $P(k)$ MUST ALSO BE TRUE.

NOTE: AS PRACTICE EXERCISES YOU SHOULD PROVE THAT THE FIRST $P(n)$ ABOVE IS TRUE FOR ALL $n \geq 2$ AND THE SECOND IS TRUE FOR ALL $n \geq 1$.

EXAMPLES:

1. ON A LARGE, FLAT FIELD, 2001 PEOPLE ARE POSITIONED SO THAT, FOR EACH PERSON, THE DISTANCES TO THE OTHER PEOPLE ARE ALL DIFFERENT. EACH PERSON HAS A WATER PISTOL AND, WHEN A SIGNAL IS SOUNDED, SHOOTS THE PERSON WHO IS CLOSEST. SHOW THAT THERE IS AT LEAST ONE PERSON WHO WILL BE LEFT DRY.

SOLUTION : NOTE THAT, IF THERE WERE ONLY 2 PEOPLE ON THE FIELD, NEITHER WOULD REMAIN DRY AND IF THERE WERE 4 PEOPLE IT MAY OR MAY NOT BE THE CASE THAT ONE REMAINS DRY, E.G.,



HOWEVER, IF THERE WERE 3 PEOPLE, ONE WOULD REMAIN DRY :

PROOF : LET P_1, P_2, P_3 DENOTE THE PEOPLE AND $d(P_1, P_2)$, $d(P_1, P_3)$, $d(P_2, P_3)$ THE DISTANCES BETWEEN THEM. THERE ARE ONLY THREE OF THEM SO ONE IS LESS THAN OR EQUAL TO THE OTHER TWO, SAY, $d(P_2, P_3)$. THEN P_2 AND P_3 SHOOT EACH OTHER AND, WHOEVER P_1 SHOOTS, NO ONE SHOOTS HIM SO HE REMAINS DRY.

WE WILL SHOW, BY INDUCTION, THAT THE SAME IS TRUE FOR ANY ODD INTEGER ≥ 3, I.E., FOR EVERY $2n+1$, $n \geq 1$.

WE JUST VERIFIED THE STATEMENT FOR $n=1$ SO NOW ASSUME THE RESULT FOR $n = k$, I.E., WHEN $2k+1$ PEOPLE PLAY THE GAME AT LEAST ONE REMAINS DRY. SUPPOSE THAT $2(k+1)+1 = 2k+3$ PEOPLE PLAY THE GAME, SAY,

$$P_1, \ldots, P_{2k+1}, P_{2k+2}, P_{2k+3} .$$

THE SET OF DISTANCES BETWEEN DISTINCT P_i AND P_j HAS A MINIMUM. CHOOSE SOME PAIR FOR WHICH THIS MINIMUM IS ACHIEVED AND SUPPOSE, WITHOUT LOSS OF GENERALITY, THAT THIS PAIR IS P_{2k+2} AND P_{2k+3}. THEN P_{2k+2} AND P_{2k+3} SHOOT EACH OTHER.

NOW CONSIDER $P_1, \ldots, P_{2k+1}$. IF NONE OF THESE SHOOT P_{2k+2} OR P_{2k+3}, THEN THE INDUCTION HYPOTHESIS IMPLIES THAT AT LEAST ONE OF $P_1, \ldots, P_{2k+1}$ REMAINS DRY. IF ONE OF THEM DOES SHOOT EITHER P_{2k+2} OR P_{2k+3}, THEN THE $2k+1$ SHOTS FIRED BY $P_1, \ldots, P_{2k+1}$ CAN HIT AT MOST $2k$ OF $P_1, \ldots, P_{2k+1}$ SO AT LEAST ONE MUST REMAIN DRY. IN ANY CASE, AT LEAST ONE OF THEM REMAINS DRY.

2. SHOW THAT EVERY POSITIVE INTEGER IS THE SUM OF ONE OR MORE NUMBERS OF THE FORM $2^r 3^s$, WHERE r AND s ARE NON-NEGATIVE INTEGERS AND NO SUMMAND DIVIDES ANOTHER.

$$\text{E.G., } 23 = 9 + 8 + 6 = 2^0 3^2 + 2^3 3^0 + 2^1 3^1$$

<u>SOLUTION</u>: BY INDUCTION. FOR $n=1$ WE HAVE $1 = 2^0 3^0$. NOW SUPPOSE THAT ALL POSITIVE INTEGERS $< k$ HAVE SUCH A REPRESENTATION. IF k IS EVEN, THEN WE TAKE SUCH A REPRESENTATION FOR $\frac{k}{2}$ AND MULTIPLY THROUGH BY 2 TO GET A REPRESENTATION FOR k. NOW SUPPOSE k IS ODD. IF k IS A POWER OF 3 WE ARE DONE ($k = 3^r = 2^0 3^r$) SO ASSUME THIS IS NOT THE CASE. CHOOSE A POSITIVE INTEGER m WITH $3^m < k < 3^{m+1}$ ($m = \lfloor \log_3 k \rfloor$). THEN $k - 3^m$ IS POSITIVE, EVEN AND $< k$ SO $\frac{k-3^m}{2}$ HAS A REPRESENTATION OF THE REQUIRED FORM, SAY

$$\frac{k-3^m}{2} = s_1 + \cdots + s_\ell$$

SO

$$k = 2s_1 + \cdots + 2s_\ell + 3^m .$$

CLEARLY, NO $2s_i$ DIVIDES 3^m OR ANY OTHER $2s_j$ SO WE NEED ONLY SHOW THAT 3^m CANNOT DIVIDE ANY $2s_i$. FOR THIS IT IS ENOUGH TO SHOW THAT 3^m CANNOT DIVIDE s_i. BUT, FOR EACH i,

$$2s_i \le k - 3^m < 3^{m+1} - 3^m = 2 \cdot 3^m$$

SO

$$s_i < 3^m$$

AND THE PROOF IS COMPLETE.

A FACT WORTH REMEMBERING :

<u>ARITHMETIC</u> - <u>GEOMETRIC</u> <u>MEAN</u> <u>INEQUALITY</u> : $x_1, \ldots, x_n \geq 0 \Rightarrow$

$$\sqrt[n]{x_1 \cdots x_n} \le \frac{x_1 + \cdots + x_n}{n}$$