PARITY

IN SOME SENSE, THE USE OF "PARITY" IS JUST A SPECIAL CASE OF MODULAR ARITHMETIC ("THINK MOD 2").

E.G., SUPPOSE SOME FAST FOOD CHAIN SELLS PACKAGES OF SIX OR TWENTY CHICKEN NUGGETS, BUT YOU WANT TO BUY EXACTLY 2137 CHICKEN NUGGETS (YOU'RE VERY HUNGRY). IS IT POSSIBLE? THE ANSWER IS CLEARLY "NO" SINCE ANY MULTIPLE OF 6 PLUS ANY MULTIPLE OF 20 IS EVEN, BUT 2137 IS NOT.

MORE FORMALLY, WE ARE LOOKING FOR INTEGERS x AND y SUCH THAT $6x + 20y = 2137$, BUT

$$6x + 20y \equiv 0x + 0y \bmod 2 \equiv 0 \bmod 2$$

WHEREAS

$$2137 \equiv 1 \bmod 2 .$$

HOWEVER, THE "EVEN/ODD" DICHOTOMY MORE OFTEN ENTERS IN A MORE SUBTLE, DISGUISED FORM, E.G., "RED/BLACK", "UP/DOWN", ETC.

EXAMPLES:

1. CONSIDER A CHESSBOARD (8 ROWS AND 8 COLUMNS OF SQUARES, ALTERNATELY RED AND BLACK). CUT OUT TWO

DIAGONALLY OPPOSITE SQUARES AND THROW THEM AWAY. THERE ARE NOW 62 SQUARES LEFT. IS IT POSSIBLE TO COVER THESE SQUARES WITH EXACTLY 31 DOMINOES? (A DOMINOE IS A RECTANGLE THAT COVERS EXACTLY 2 SQUARES)

DIAGONALLY OPPOSITE SQUARES MUST HAVE THE SAME COLOR (SAY, BLACK). THUS, THE REMAINING 62 SQUARES CONSIST OF 32 RED SQUARES AND 30 BLACK SQUARES. EACH DOMINOE COVERS ONE SQUARE OF EACH COLOR SO 31 DOMINOES WILL COVER EXACTLY 31 RED SQUARES AND WE CANNOT COVER ALL 32.

<u>NOTE</u>: THE PROBLEM WOULD HAVE BEEN MORE INTERESTING IF THE SQUARES HAD NOT BEEN COLORED. WE WOULD HAVE HAD TO THINK TO DO IT OURSELVES.

2. THERE ARE 25 COINS ON A TABLE, ALL "HEADS UP". YOU ARE PERMITTED TO FLIP OVER EXACTLY TWO AT A TIME, AS MANY TIMES AS YOU LIKE. CAN YOU CHANGE ALL OF THEM TO "HEADS DOWN"?

CALL FLIPPING TWO OF THE COINS A "MOVE". WE WILL SHOW THAT, FOR ANY $n \geq 1$, AFTER n MOVES THE NUMBER OF COINS "HEADS UP" MUST BE ODD (AND SO CANNOT BE 0). THE PROOF IS BY INDUCTION. FOR $n = 1$ THIS IS CLEAR SINCE WE MUST FLIP TWO HEADS UP COINS TO HEADS DOWN, LEAVING 23 STILL UP. NOW SUPPOSE $k \geq 1$ AND THAT AFTER ANY k MOVES THE NUMBER OF COINS HEADS UP IS ODD. CONSIDER A SEQUENCE OF $k+1$ MOVES. THE FIRST k MOVES LEAVE US WITH AN ODD NUMBER OF COINS HEADS UP, SAY, $2\ell+1$. THE $(k+1)^{ST}$ MOVE EITHER FLIPS TWO UP TO DOWN, TWO DOWN TO UP, OR ONE DOWN TO UP AND ONE UP TO DOWN SO THE NUMBER UP IS EITHER $2\ell-1$, $2\ell+3$, OR $2\ell+1$, ALL OF WHICH ARE ODD.

3. LET $n > 1$ BE AN ODD INTEGER. LET A BE AN $n \times n$ SYMMETRIC MATRIX SUCH THAT EACH ROW OF A IS SOME PERMUTATION OF $1, \ldots, n$. SHOW THAT EACH ONE OF $1, \ldots, n$ MUST APPEAR IN THE MAIN DIAGONAL OF A.

SINCE EVERY ROW IS A PERMUTATION OF $1, \ldots, n$, EACH NUMBER IN $\{1, \ldots, n\}$ APPEARS EXACTLY n TIMES IN A. BY SYMMETRY THE NUMBERS OCCURRING AS OFF-DIAGONAL ENTRIES OCCUR IN PAIRS AND SO APPEAR AN EVEN NUMBER OF TIMES OFF-DIAGONAL. n ODD $\Rightarrow$ EACH SUCH ELEMENT APPEARS AT LEAST ONCE ON THE DIAGONAL. BUT ANY ELEMENT THAT DOES NOT APPEAR OFF-DIAGONAL HAS TO BE ON THE DIAGONAL. THUS, EVERY ELEMENT OF $\{1, \ldots, n\}$ APPEARS ON THE DIAGONAL.