PIGEONHOLE PRINCIPLE

" AMONG ANY 13 PEOPLE, AT LEAST TWO ARE BORN IN THE SAME MONTH."

12 PIGEONHOLES (MONTHS) ; 13 PIGEONS (PEOPLE)

" AMONG ANY 29 PEOPLE, AT LEAST FIVE WERE BORN ON THE SAME DAY OF THE WEEK."

7 PIGEONHOLES (DAYS OF THE WEEK) ; $29 = 4 \cdot 7 + 1$ PIGEONS (PEOPLE)

THAT'S ALL THERE IS TO IT.

IF $n+1$ (OR MORE) PIGEONS ARE PLACED IN n PIGEONHOLES,
THEN AT LEAST ONE OF THE PIGEONHOLES CONTAINS TWO OR MORE PIGEONS.

OR, MORE GENERALLY,

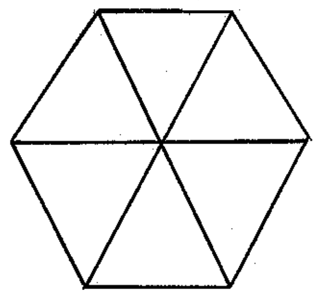
IF $kn+1$ (OR MORE) PIGEONS ARE PLACED IN n PIGEONHOLES,
THEN AT LEAST ONE OF THE PIGEONHOLES CONTAINS $k+1$ OR MORE PIGEONS.

KNOWING WHEN AND HOW TO APPLY THIS SIMPLE-MINDED IDEA IS NOT ALWAYS SO EASY, HOWEVER.

EXAMPLES:

1. SELECT ANY 7 POINTS ON OR INSIDE A REGULAR HEXAGON OF SIDE LENGTH 1. SHOW THAT THERE ARE TWO POINTS AT MOST ONE UNIT APART.

CONNECTING OPPOSITE VERTICES OF THE HEXAGON GIVES SIX EQUILATERAL TRIANGLES OF SIDE LENGTH 1 (ALL INTERIOR ANGLES ARE 60°).



6 PIGEONHOLES (TRIANGLES) ; 7 PIGEONS (SELECTED POINTS)

AT LEAST TWO OF THE POINTS MUST LIE IN THE SAME TRIANGLE AND THE MAXIMUM DISTANCE BETWEEN ANY TWO POINTS IN THE SAME TRIANGLE IS 1.

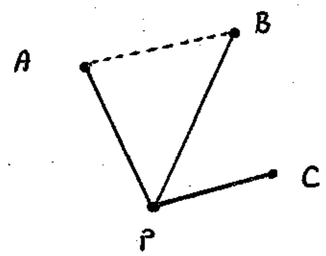
2. CONSIDER SIX POINTS IN THE PLANE, NO THREE OF WHICH ARE COLLINEAR. DRAW THE LINE SEGMENTS THAT CONNECT EVERY POINT TO EVERY OTHER POINT ($\binom{6}{2} = 15$ LINE SEGMENTS ALTOGETHER) AND COLOR EACH LINE SEGMENT EITHER RED OR BLUE (ANYWAY YOU LIKE). SHOW THAT THERE MUST BE A MONOCHROMATIC TRIANGLE.

SELECT ONE OF THE POINTS P. THERE ARE 5 EDGES FROM P TO THE REMAINING POINTS.

2 PIGEONHOLES (COLORS) ; $5 = 2 \cdot 2 + 1$ PIGEONS (EDGES FROM P)

AT LEAST 3 OF THESE EDGES HAVE THE SAME COLOR (SAY, RED). CALL THEM PA, PB, AND PC.

IF AB IS RED, THEN THE TRIANGLE PAB IS RED AND WE ARE DONE. SUPPOSE THEN THAT AB IS BLUE.



IF AC AND BC ARE ALSO BLUE, THEN ABC IS BLUE AND WE ARE DONE. OTHERWISE, EITHER AC IS RED, IN WHICH CASE PAC IS RED, OR BC IS RED, IN WHICH CASE PBC IS RED. IN ANY CASE, THERE IS A MONOCHROMATIC TRIANGLE.

3. SUPPOSE THAT THERE ARE SIX PEOPLE AT A PARTY. SHOW THAT, AMONG THE SIX THERE ARE EITHER THREE MUTUAL ACQUAINTANCES OR THREE MUTUAL STRANGERS.

REPRESENT THE SIX PEOPLE BY SIX POINTS IN THE PLANE, NO THREE OF WHICH ARE COLLINEAR. DRAW LINE SEGMENTS JOINING EACH PAIR OF PEOPLE AND COLOR THE SEGMENT RED IF THE TWO PEOPLE ARE ACQUAINTED AND BLUE IF THEY ARE STRANGERS. THIS IS NOW EXAMPLE 2 AND A MONOCHROMATIC TRIANGLE CORRESPONDS TO THREE PEOPLE (VERTICES) THAT ARE EITHER MUTUAL ACQUAINTANCES (RED) OR MUTUAL STRANGERS (BLUE).

4. GIVEN A SET OF $n+1$ POSITIVE INTEGERS, NONE OF WHICH EXCEEDS $2n$, SHOW THAT AT LEAST ONE MEMBER OF THE SET MUST DIVIDE ANOTHER MEMBER OF THE SET.

THE KEY IS THAT THERE ARE EXACTLY n ODD POSITIVE INTEGERS $\leq 2n$.

LET $x_1, \ldots, x_{n+1}$ BE THE POSITIVE INTEGERS $\leq 2n$ AND WRITE

$$x_i = 2^{n_i} y_i \quad , \quad i = 1, \ldots, n+1,$$

WHERE EACH n_i IS A NON-NEGATIVE INTEGER AND EACH y_i IS ODD.

n PIGEONHOLES (ODD POSITIVE INTEGERS $\leq 2n$); $n+1$ PIGEONS ($y_1, \ldots, y_{n+1}$)

FOR SOME $i_0, j_0 = 1, \ldots, n+1$, $y_{i_0} = y_{j_0}$ SO

$$x_{i_0} = 2^{n_{i_0}} y_{i_0} \qquad \text{AND} \qquad x_{j_0} = 2^{n_{j_0}} y_{i_0} .$$

IF $n_{i_0} \leq n_{j_0}$, THEN x_{i_0} DIVIDES x_{j_0}; IF $n_{i_0} > n_{j_0}$, THEN x_{j_0} DIVIDES x_{i_0}.

OUR FINAL EXAMPLE ALSO LEADS NATURALLY INTO OUR NEXT TOPIC (MODULAR ARITHMETIC).

5. SHOW THAT THERE MUST EXIST A POSITIVE INTEGER CONTAINING ONLY THE DIGITS 0 AND 1 THAT IS DIVISIBLE BY 1999.

CONSIDER THE SET OF THE FIRST 2000 NUMBERS OF THE FORM

$$1, \; 11, \; 111, \; 1111, \ldots \qquad \text{(2000 OF THEM)}.$$

WHEN ANY OF THESE NUMBERS IS DIVIDED BY 1999 THE REMAINDER IS ONE OF

$$0, 1, 2, 3, \ldots, 1998 \quad \text{(1999 OF THEM)}$$

THUS, AT LEAST TWO OF THESE NUMBERS HAVE THE SAME REMAINDER WHEN DIVIDED BY 1999. LET a AND b BE TWO OF THESE AND ASSUME $a < b$. THEN $b-a$ IS DIVISIBLE BY 1999 AND ITS DIGITS CAN BE ONLY 0 OR 1 BECAUSE a AND b HAVE ONLY THE DIGIT 1.

A FACT WORTH REMEMBERING:

a, b REAL NUMBERS AND n A POSITIVE INTEGER $\Rightarrow$

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

IF n IS ODD,

$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \cdots - ab^{n-2} + b^{n-1})$$