PROBLEM SOLVING SEMINAR

PROBLEM POTPOURRI

1. SHOW THAT

$$\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \cdots + (-1)^{n+1}\frac{1}{n}\binom{n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

2. LET S BE A SET OF REAL NUMBERS THAT IS CLOSED UNDER MULTIPLICATION (THAT IS, IF a AND b ARE IN S, THEN SO IS ab). LET T AND U BE DISJOINT SUBSETS OF S WHOSE UNION IS S. GIVEN THAT THE PRODUCT OF ANY THREE (NOT NECESSARILY DISTINCT) ELEMENTS OF T IS IN T AND THAT THE PRODUCT OF ANY THREE ELEMENTS OF U IS IN U, SHOW THAT AT LEAST ONE OF THE TWO SUBSETS T, U IS CLOSED UNDER MULTIPLICATION.

3. CAN UNCOUNTABLY MANY NONINTERSECTING COPIES OF THE FIGURE-EIGHT WITH WHATEVER ORIENTATION AND SIZE (E.G., 8, ∞, 8, ETC.) BE PACKED INTO THE PLANE? JUSTIFY YOUR ANSWER.

4. DEFINE A SEQUENCE $\{u_n\}_{n=0}^{\infty}$ BY $u_0 = u_1 = u_2 = 1$ AND, FOR $n \geq 0$,

$$\det\begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!\,.$$

SHOW THAT u_n IS AN INTEGER FOR ALL $n \geq 0$.

5. A disk of radius 1 cm. has a small hole at a point half way between the center and the circumference. The disk is lying inside a circle of radius 2 cm. A pen is put through the hole in the disk, and then the disk is moved once round the inside of the circle, keeping the disk in contact with the circle without slipping, so the pen draws a curve. What is the area enclosed by the curve?

6. LET $f: \mathbb{R} \to \mathbb{R}$ BE TWICE DIFFERENTIABLE AND SATISFY

$$f(x) + f''(x) = -x\,g(x)\,f'(x)$$

WHERE $g(x) \geq 0$ FOR ALL REAL X. PROVE THAT $|f(x)|$ IS BOUNDED.

7. BASKETBALL STAR SHANILLE O'KEAL'S TEAM STATISTICIAN KEEPS TRACK OF THE NUMBER $S(N)$ OF SUCCESSFUL FREE THROWS SHE HAS MADE IN HER FIRST N ATTEMPTS OF THE SEASON. EARLY IN THE SEASON $S(N)$ WAS LESS THAN 80% OF N, BUT BY THE END OF THE SEASON, $S(N)$ WAS GREATER THAN 80% OF N. WAS THERE NECESSARILY A MOMENT IN BETWEEN WHEN $S(N)$ WAS EXACTLY 80% OF N? EXPLAIN.

8. LET $P(x)$ BE A POLYNOMIAL WITH INTEGER COEFFICIENTS EACH OF WHICH IS NO LARGER THAN K IN ABSOLUTE VALUE. SUPPOSE $P(x) = 0$ FOR SOME x WITH $|x| > K+1$. FIND $P(x)$.

9. A <u>LATTICE POINT</u> IN THE PLANE IS A POINT WHOSE COORDINATES ARE BOTH INTEGERS. CHOOSE ANY FIVE DISTINCT LATTICE POINTS. SHOW THAT TWO OF THESE POINTS HAVE THE PROPERTY THAT SOME POINT ON THE INTERIOR OF THE LINE SEGMENT JOINING THEM IS A LATTICE POINT.

10. LET α BE A REAL NUMBER. FOR EACH INTEGER $m \geq 0$ DEFINE A SEQUENCE $(a_m(j))_{j=0}^{\infty}$ BY

$$a_m(0) = \frac{\alpha}{2^m}$$

$$a_m(j+1) = (a_m(j))^2 + 2a_m(j), \quad j \geq 0.$$

EVALUATE

$$\lim_{n \to \infty} a_n(n)$$

11. A CROSS-COUNTRY RUNNER FINISHES A SIX-MILE COURSE IN 30 MINUTES. PROVE THAT SOMEWHERE ALONG THE COURSE SHE RAN ONE MILE IN EXACTLY 5 MINUTES. YOU WILL NEED TO MAKE SOME PHYSICALLY REASONABLE CONTINUITY ASSUMPTION.

12. DETERMINE, WITH PROOF, THE NUMBER OF ORDERED TRIPLES (A_1, A_2, A_3) OF SETS SATISFYING

(i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$.

13. A LARGE NUMBER OF SPY SATELLITES ORBIT THE EARTH; THEIR PRECISE NUMBER IS A MILITARY SECRET. THEY COMMUNICATE CONTINUOUSLY WITH EACH OTHER EXCEPT WHEN THE EARTH INTERRUPTS THE LINE-OF-SIGHT PATH BETWEEN THEM. PROVE THAT AT ALL TIMES AT LEAST TWO SATELLITES ARE EACH IN UNINTERRUPTED COMMUNICATION WITH THE SAME NUMBER OF SATELLITES (PERHAPS ZERO).

14. LET $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ BE A REAL POLYNOMIAL OF DEGREE $n \geq 2$ SUCH THAT

$$0 < a_0 < - \sum_{j=1}^{[\frac{n}{2}]} \frac{1}{2j+1} a_{2j}$$

($[\frac{n}{2}]$ IS THE GREATEST INTEGER $\leq \frac{n}{2}$). SHOW THAT $P(x)$ HAS A REAL ZERO IN $(-1, 1)$.

15. **Let n be a positive integer. Which is larger, n^{n+1} or $(n+1)^n$?**

16. LET $n \geq 2$ BE AN INTEGER AND T_n BE THE NUMBER OF NONEMPTY SUBSETS S OF $\{1, 2, 3, \ldots, n\}$ WITH THE PROPERTY THE AVERAGE OF THE ELEMENTS IN S IS AN INTEGER. PROVE THAT $T_n - n$ IS ALWAYS EVEN.

17. LET d_n BE THE DETERMINANT OF THE $n \times n$ MATRIX WHOSE ENTRIES, FROM LEFT TO RIGHT AND THEN FROM TOP TO BOTTOM, ARE $\cos 1, \cos 2, \ldots, \cos n^2$. (FOR EXAMPLE,

$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix} .$$

THE ARGUMENT OF COS IS ALWAYS IN RADIANS, NOT DEGREES.) EVALUATE $\lim_{n \to \infty} d_n$.

18. PROVE THAT THERE EXIST AN INFINITE NUMBER OF ORDERED PAIRS (a, b) OF INTEGERS SUCH THAT, FOR EVERY POSITIVE INTEGER t, THE NUMBER $at+b$ IS A TRIANGULAR NUMBER IF AND ONLY IF t IS A TRIANGULAR NUMBER. (THE TRIANGULAR NUMBERS ARE THE $t_n = \frac{n(n+1)}{2}$ WITH n IN $\{0, 1, 2, \ldots\}$.)

19. A COMMON CALCULUS MISTAKE IS TO BELIEVE THAT THE PRODUCT RULE FOR DERIVATIVES SAYS THAT $(fg)' = f'g'$. IF $f(x) = e^{x^2}$, DETERMINE, WITH PROOF, WHETHER THERE EXISTS AN OPEN INTERVAL (a, b) AND A NONZERO FUNCTION g DEFINED ON (a, b) SUCH THAT THIS WRONG PRODUCT RULE IS TRUE FOR x IN (a, b).

20. THE FUNCTION $K(x, y)$ IS POSITIVE AND CONTINUOUS FOR $0 \le x \le 1$, $0 \le y \le 1$ AND THE FUNCTIONS $f(x)$ AND $g(x)$ ARE POSITIVE AND CONTINUOUS FOR $0 \le x \le 1$. SUPPOSE THAT FOR ALL x, $0 \le x \le 1$,

$$\int_0^1 f(y) K(x, y)\, dy = g(x) \quad \text{AND} \quad \int_0^1 g(y) K(x, y)\, dy = f(x).$$

SHOW THAT $f(x) = g(x)$ FOR $0 \le x \le 1$.

21. GIVEN A POSITIVE INTEGER n, IN HOW MANY WAYS CAN WE WRITE n AS A SUM OF ONE OR MORE POSITIVE INTEGERS

$$a_1 \le \cdots \le a_k$$

WHERE $a_k - a_1 = 0$ OR 1 ?

22. WHICH IS LARGER, $99^{49} + 100^{49}$ OR 101^{49} ?

23. MATHEMATICS DEPARTMENTS AT SOME SOUTH-WESTERN UNIVERSITIES RECEIVED MR. H. N.'S MISCHIEVOUS LETTER ASKING FOR THE ONE REAL SOLUTION X TO THE TWO EQUATIONS

$$\frac{(1+X)^{17}}{X} = 17$$

AND

$$\frac{(1+x)^{18}}{x} = 18.$$

PROFESSOR A.S. AT ONE UNIVERSITY SENT MR. H. N. THE FOLLOWING BRIEF SOLUTION :

$$\frac{18}{17} = \frac{(1+x)^{18}/x}{(1+x)^{17}/x} = 1+X$$

SO

$$X = \frac{18}{17} - 1 = \frac{1}{17}.$$

IS THIS THE ONLY SOLUTION ? EXPLAIN !

24. AN ELLIPSE, WHOSE SEMI-AXES HAVE LENGTHS a AND b, ROLLS WITHOUT SLIPPING ON THE CURVE $y = c \sin\left(\frac{x}{a}\right)$. HOW ARE a, b AND c RELATED, GIVEN THAT THE ELLIPSE COMPLETES ONE REVOLUTION WHEN IT TRAVERSES ONE PERIOD OF THE CURVE ?

25. FOR WHAT PAIRS (a,b) OF POSITIVE REAL NUMBERS DOES THE IMPROPER INTEGRAL

$$\int_b^\infty \left(\sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right) dx$$

CONVERGE ?

26. LET G BE A (MULTIPLICATIVE) GROUP AND DENOTE ITS UNIT ELEMENT BY 1. LET p BE A PRIME. PROVE THAT p DIVIDES THE NUMBER OF SOLUTIONS TO

$$x^p = 1$$

UNLESS THAT NUMBER IS INFINITE OR 1.

27. PROVE THAT, FOR $n \geq 2$,

$$\overbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}^{n \text{ TERMS}} \equiv \overbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}^{n-1 \text{ TERMS}} \mod n$$