PROBLEM SOLVING SEMINAR

PROBLEM SET 1 (SOLUTIONS)

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1. LET $f$ BE DEFINED ON THE NATURAL NUMBERS BY $f(1) = 1$ AND, FOR $n > 1$,

$$f(n) = f(f(n-1)) + f(n - f(n-1)).$$

FIND, WITH PROOF, A SIMPLE EXPLICIT EXPRESSION FOR $f(n)$.

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NOTICE THAT

$$\begin{aligned}
f(1) &= 1 \\
f(2) &= f(f(1)) + f(2 - f(1)) \\
&= f(1) + f(1) = 1+1 \\
&= 2 \\
f(3) &= f(f(2)) + f(3 - f(2)) \\
&= f(2) + f(1) = 2+1 \\
&= 3
\end{aligned}$$

SO WE WILL CONJECTURE THAT $f(n) = n$ FOR ALL $n \geq 1$. WE PROVE THIS BY INDUCTION. THIS HAS ALREADY BEEN CHECKED FOR $n = 1, 2, 3$ SO WE ASSUME $f(n) = n$ FOR ALL $1 \leq n < k$ AND SHOW FROM THIS THAT $f(k) = k$. BUT

$$\begin{aligned}
f(k) &= f(f(k-1)) + f(k - f(k-1)) \\
&= f(k-1) + f(k - (k-1)) \\
&= (k-1) + f(1) = (k-1)+1 \\
&= k
\end{aligned}$$

SO THE RESULT FOLLOWS.

2. A DART, THROWN AT RANDOM, HITS A SQUARE TARGET. ASSUMING THAT ANY TWO PARTS OF THE TARGET OF EQUAL AREA ARE EQUALLY LIKELY TO BE HIT, FIND THE PROBABILITY THAT THE POINT HIT IS NEARER THE CENTER THAN TO ANY EDGE.

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OVERLAY A COORDINATE SYSTEM ON THE TARGET SO THAT THE VERTICES OF THE TARGET ARE AT $(1,1)$, $(-1,-1)$, $(-1,1)$ AND $(1,-1)$. LET $S$ BE THE SET OF POINTS THAT ARE CLOSER TO THE ORIGIN THAN TO ANY EDGE. THEN THE PROBABILITY IS

$$\frac{\text{AREA OF } S}{\text{AREA OF TARGET}} = \frac{1}{4}(\text{AREA OF } S)$$

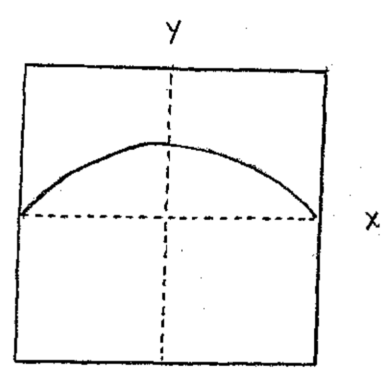
WE DETERMINE $S$ AS FOLLOWS:

$(x,y)$ IS CLOSER TO $(0,0)$ THAN TO THE TOP EDGE $y=1$ $\Longleftrightarrow$

$$\sqrt{x^2+y^2} < 1-y$$

$$\Longleftrightarrow \quad x^2+y^2 < 1-2y+y^2$$

$$\Longleftrightarrow \quad y < \tfrac{1}{2}(1-x^2)$$ (POINTS UNDER THE PARABOLA)

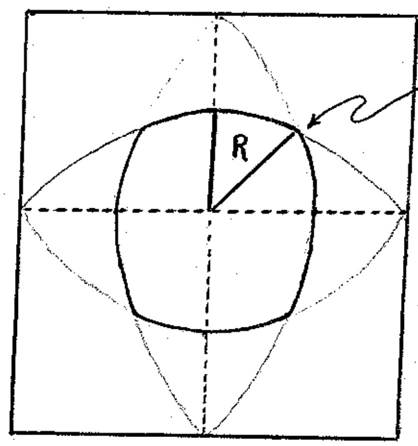


SIMILARLY FOR THE OTHER EDGES :

$$y = -1 \quad : \quad y > \tfrac{1}{2}(x^2 - 1)$$

$$x = 1 \quad : \quad x < \tfrac{1}{2}(1 - y^2)$$

$$x = -1 \quad : \quad x > \tfrac{1}{2}(y^2 - 1)$$




$y = \frac{1}{2}(1-x^2)$ AND $x = \frac{1}{2}(1-y^2)$

INTERSECT WHEN $y = x$ AND THERE

$$x = \tfrac{1}{2}(1-x^2)$$

$$x^2 + 2x - 1 = 0$$

$$x = -1 \pm \sqrt{2}$$

BY SYMMETRY, AREA OF $S$ = 8 (AREA OF $R$)

$$\text{AREA OF } R = \int_0^{-1+\sqrt{2}} \int_x^{\frac{1}{2}(1-x^2)} dy\, dx = \int_0^{-1+\sqrt{2}} \left(\tfrac{1}{2}(1-x^2) - x\right) dx$$

$$= \tfrac{1}{2}x - \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 \Big|_0^{-1+\sqrt{2}} = \frac{4\sqrt{2} - 5}{6}$$

SO

$$\text{PROBABILITY} = \tfrac{1}{4}\left(8\left(\frac{4\sqrt{2}-5}{6}\right)\right) = \frac{4\sqrt{2}-5}{3}$$

3. WHICH OF $\int_0^{\pi} \exp(\sin^2 x)\,dx$ AND $\frac{3\pi}{2}$ IS LARGER AND WHY ?

FOR $t > 0$,

$$\exp(t) = 1 + t + \tfrac{1}{2}t^2 + \cdots > 1 + t$$

SO

$$\exp(\sin^2 x) > 1 + \sin^2 x$$

FOR $0 < x < \pi$. THUS,

$$\int_0^{\pi} \exp(\sin^2 x)\,dx > \int_0^{\pi} 1 + \sin^2 x\,dx = \int_0^{\pi} 1 + \tfrac{1}{2} - \tfrac{1}{2}\cos 2x\,dx$$

$$= \tfrac{3}{2}x \Big|_0^{\pi} - \tfrac{1}{4}\sin 2x \Big|_0^{\pi}$$

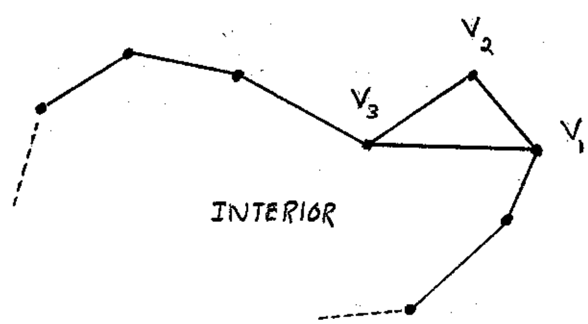
$$= \frac{3\pi}{2}$$

4. PROVE THAT THE SUM OF THE INTERIOR ANGLES IN A POLYGON WITH 1001 SIDES IS $999\pi$.

INSTEAD WE WILL PROVE, BY INDUCTION, THAT THE SUM OF THE INTERIOR ANGLES IN A POLYGON WITH $n$ SIDES, $n \geq 3$, IS $(n-2)\pi$. THEN $n = 1001$ GIVES THE RESULT.

FOR $n = 3$ THIS IS SIMPLY THE KNOWN FACT THAT THE SUM OF THE INTERIOR ANGLES IN A TRIANGLE IS $\pi$.

NOW ASSUME THAT, FOR SOME $k \geq 3$, THE SUM OF THE INTERIOR ANGLES IN A POLYGON WITH $k$ SIDES IS $(k-2)\pi$. CONSIDER A POLYGON WITH $k+1$ SIDES. CHOOSE THREE CONSECUTIVE VERTICES $V_1$, $V_2$ AND $V_3$ AND JOIN $V_1$ AND $V_3$ WITH A LINE SEGMENT.



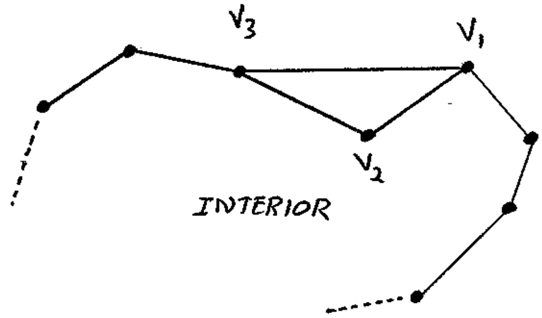

IF THIS LINE SEGMENT IS IN THE INTERIOR OF THE POLYGON THEN IT PARTITIONS THIS POLYGON INTO A POLYGON WITH $k$ SIDES AND A TRIANGLE.

BY THE INDUCTION HYPOTHESIS, THE SUM OF THE INTERIOR ANGLES OF THE $k$-GON IS $(k-2)\pi$. THE SUM OF THE INTERIOR ANGLES OF THE TRIANGLE IS $\pi$. THUS, THE SUM OF THE INTERIOR ANGLES OF THE $(k+1)$-GON IS

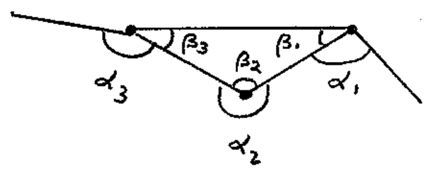
$$(k-2)\pi + \pi = (k-1)\pi = ((k+1)-2)\pi$$

AS REQUIRED.

IF THE LINE SEGMENT JOINING $V_1$ AND $V_3$ IS IN THE EXTERIOR OF THE POLYGON




THEN PROCEED AS FOLLOWS:




CONSIDER THE $k$-GON OBTAINED BY DELETING THE NEWLY FORMED TRIANGLE. BY THE INDUCTION HYPOTHESIS THE SUM OF ITS INTERIOR ANGLES IS $(k-2)\pi$. THUS, THE SUM OF THE INTERIOR ANGLES OF THE ORIGINAL $(k+1)$-GON IS

$$
\begin{aligned}
(k-2)\pi - \beta_1 - \beta_3 + 2\pi - \beta_2 &= k\pi - 2\pi - \beta_1 - \beta_3 + 2\pi - (\pi - (\beta_1 + \beta_3)) \\
&= k\pi - \pi \\
&= (k-1)\pi \\
&= ((k+1) - 2)\pi
\end{aligned}
$$

AS REQUIRED.

5. DOES THERE EXIST AN INFINITE SEQUENCE OF CLOSED DISCS $D_1, D_2, D_3, \ldots$ IN THE PLANE WITH CENTERS $c_1, c_2, c_3, \ldots$, RESPECTIVELY, SUCH THAT

(i) THE $c_i$ HAVE NO LIMIT POINT IN THE PLANE,

(ii) THE SUM OF THE AREAS OF THE $D_i$ IS FINITE, AND

(iii) EVERY LINE IN THE PLANE INTERSECTS AT LEAST ONE OF THE $D_i$.

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EVERY LINE IN THE PLANE INTERSECTS ONE OF THE COORDINATE AXES SO IT WILL SUFFICE TO COVER THESE WITH A FAMILY OF DISCS SATISFYING (i) AND (ii). IF $r_i$ IS THE RADIUS OF $D_i$, THEN (ii) REQUIRES THAT $\sum_{i=1}^{\infty} r_i^2 < \infty$. WE WILL TAKE THE $r_i$ TO BE ANY SEQUENCE OF POSITIVE NUMBERS FOR WHICH $\sum_{i=1}^{\infty} r_i = \infty$, BUT $\sum_{i=1}^{\infty} r_i^2 < \infty$, E.G., $r_i = \frac{1}{i}$ FOR $i \geq 1$. THE PARTIAL SUMS $S_i = r_1 + \cdots + r_i$ INCREASE TO $\infty$ AS $i \to \infty$ SO ANY NONNEGATIVE REAL NUMBER $X$ IS IN ONE OF THE INTEVALS $[0, S_1]$, $[S_1, S_2], \ldots, [S_{i-1}, S_i], \ldots$ THESE HAVE LENGTHS $r_1, r_2, \ldots, r_i, \ldots$ SO $S_{i-1} \leq X \leq S_i \Rightarrow -r_i \leq X - S_i \leq 0 \Rightarrow |X - S_i| \leq r_i$ FOR SOME $i$. CONSEQUENTLY, THE CLOSED DISCS OF RADIUS $r_i$ CENTERED AT $(S_i, 0)$, $i = 1, 2, \ldots$ COVER THE NONNEGATIVE $X$-AXIS. THOSE OF RADIUS $r_i$ ABOUT $(-S_i, 0)$, $(0, S_i)$, AND $(0, -S_i)$ COVER THE REMAINING AXES. THE SUM OF THE AREAS OF THESE DISCS IS FINITE BECAUSE $\sum_{i=1}^{\infty} r_i^2 < \infty$ AND THE CENTERS HAVE NO LIMIT POINT IN THE PLANE BECAUSE THE $S_i$ INCREASE TO $\infty$.