PROBLEM SOLVING SEMINAR

PROBLEM SET 2 (SOLUTIONS)

---

1. GIVEN ANY SET OF 10 POSITIVE INTEGERS BETWEEN 1 AND 99 (INCLUSIVE) SHOW THAT THERE ARE TWO DISJOINT (NONEMPTY) SUBSETS WITH THE SAME SUM.

LET $S = \{n_1, \ldots, n_{10}\}$ BE ANY 10-ELEMENT SUBSET OF $\{1, \ldots, 99\}$. THERE ARE $2^{10} - 1 = 1023$ NONEMPTY SUBSETS OF S. THE SUM OF THE ELEMENTS IN SUCH A SUBSET IS ≥ 1 AND ≤ 945 (THE SMALLEST SUM OF SUCH A SUBSET IS 1 AND THE LARGEST IS $90 + 91 + \cdots + 99 = 945$).

PIGEONS (NONEMPTY SUBSETS OF S)	PIGEONHOLES (SUMS)
1023	945

AT LEAST TWO DISTINCT NONEMPTY SUBSETS S_1 AND S_2 OF S MUST HAVE THE SAME SUM. IF $S_1 \cap S_2 \neq \phi$, THEN $S_1 - S_1 \cap S_2$ AND $S_2 - S_1 \cap S_2$ ARE DISJOINT, NONEMPTY AND STILL HAVE THE SAME SUM.

---

2. LET $n > 0$ BE AN INTEGER. SHOW THAT FOR ANY POSITIVE REAL NUMBER X,

$$\frac{x^n}{(x+1)^{n+1}} \leq \frac{n^n}{(n+1)^{n+1}}$$

LET $f(x) = \dfrac{x^n}{(x+1)^{n+1}}$. THEN f IS DIFFERENTIABLE ON $x > 0$ AND

$$f'(x) = \frac{(x+1)^{n+1} n x^{n-1} - x^n (n+1)(x+1)^n}{(x+1)^{2n+2}}$$

$$= \frac{x^{n-1}(x+1)^n \left(n(x+1) - x(n+1) \right)}{(x+1)^{2n+2}}$$

$$= \frac{x^{n-1}}{(x+1)^{n+2}} (n-x).$$

THE ONLY CRITICAL POINT ON $x > 0$ IS AT $x = n$. MOREOVER, $f'(x) > 0$ FOR $0 < x < n$ AND $f'(x) < 0$ FOR $x > n$ SO $f(x)$ HAS A MAXIMUM AT $x = n$, I.E.,

$$\frac{x^n}{(x+1)^{n+1}} \le \frac{n^n}{(n+1)^{n+1}}$$

FOR ALL $x > 0$.

---

3. LET $\mathbb{R}^2$ DENOTE THE XY-PLANE AND DEFINE $F : \mathbb{R}^2 \to \mathbb{R}^2$ BY

$$F(x,y) = (4x - 3y + 1,\ 2x - y + 1).$$

DETERMINE $F^{100}(1,0)$, WHERE F^{100} MEANS APPLY F 100 TIMES.

WRITE F AS

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

THEN

$$F^2\begin{pmatrix} x \\ y \end{pmatrix} = F\left(F\begin{pmatrix} x \\ y \end{pmatrix}\right) = F\left(\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}^2\begin{pmatrix} x \\ y \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

AND, BY INDUCTION,

$$F^{100}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}^{100}\begin{pmatrix} x \\ y \end{pmatrix} + 100\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

THUS,

$$F^{100}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}^{100}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 100\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

TO COMPUTE $\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}^{100}$ WE DIAGONALIZE $A = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$. THE CHARACTERISTIC EQUATION IS

$$0 = \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{vmatrix}$$

$$= (4-\lambda)(-1-\lambda) + 6$$

$$= \lambda^2 - 3\lambda + 2$$

$$= (\lambda - 1)(\lambda - 2)$$

SO THE EIGENVALUES ARE $\lambda = 1, 2$. AN EIGENVECTOR FOR $\lambda = 1$ IS

$$u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

SINCE

$$\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (1)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

AND AN EIGENVECTOR FOR $\lambda = 2$ IS

$$u_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

SINCE

$$\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

SET $P = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$ AND $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. THEN $P^{-1} = \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}$

AND

$$P^{-1}AP = D$$

SO

$$A = PDP^{-1}.$$

THUS,

$$A^{100} = PD^{100}P^{-1} = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^{100} \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3\cdot 2^{100} \\ 1 & 2\cdot 2^{100} \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2+3\cdot 2^{100} & 3-3\cdot 2^{100} \\ -2+2\cdot 2^{100} & 3-2\cdot 2^{100} \end{pmatrix}.$$

CONSEQUENTLY,

$$F^{100}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2+3\cdot 2^{100} & 3-3\cdot 2^{100} \\ -2+2\cdot 2^{100} & 3-2\cdot 2^{100} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 100\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2+3\cdot 2^{100} \\ -2+2\cdot 2^{100} \end{pmatrix} + \begin{pmatrix} 100 \\ 100 \end{pmatrix}$$

$$= \begin{pmatrix} 98+3\cdot 2^{100} \\ 98+2\cdot 2^{100} \end{pmatrix}.$$

4. GIVEN ANY FIVE POINTS ON A SPHERE SHOW THAT SOME FOUR OF THEM MUST LIE ON A CLOSED HEMISPHERE.

LET THE POINTS BE P_1, P_2, P_3, P_4, P_5. THE PLANE CONTAINING P_1, P_2 AND THE CENTER OF THE SPHERE INTERSECTS THE SPHERE IN A GREAT CIRCLE (DIAMETER). BY THE PIGEONHOLE PRINCIPLE, TWO OF THE REMAINING THREE POINTS P_3, P_4, P_5 MUST LIE ON THE SAME HEMISPHERE DETERMINED BY THIS DIAMETER. THESE TWO, TOGETHER WITH P_1 AND P_2 ARE FOUR POINTS ON THE SAME CLOSED HEMISPHERE.

5. WHICH IS LARGER, $99^{50} + 100^{50}$ OR 101^{50}? PROVE THAT YOUR ANSWER IS CORRECT.

WE COMPUTE

$$101^{50} - 99^{50} = (100+1)^{50} - (100-1)^{50}$$

$$= \sum_{n=0}^{50} \binom{50}{n} 100^n - \sum_{n=0}^{50} \binom{50}{n} 100^n (-1)^{50-n}$$

(BINOMIAL THEOREM)

$$= \sum_{n=0}^{50} \binom{50}{n} 100^n - \sum_{n=0}^{50} \binom{50}{n} 100^n (-1)^n$$

$$= 2 \sum_{k=0}^{24} \binom{50}{2k+1} 100^{2k+1}$$

$$= 2 \binom{50}{49} 100^{49} + 2 \sum_{k=0}^{23} \binom{50}{2k+1} 100^{2k+1}$$

$$= 2 \cdot 50 \cdot 100^{49} + 2 \sum_{k=0}^{23} \binom{50}{2k+1} 100^{2k+1}$$

$$= 100^{50} + 2 \sum_{k=0}^{23} \binom{50}{2k+1} 100^{2k+1}$$

$$> 100^{50}$$

so

$$101^{50} > 99^{50} + 100^{50}$$