PROBLEM SOLVING SEMINAR

PROBLEM SET 3 (SOLUTIONS)

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1. **PROVE THAT $4^{3n+1} + 2^{3n+1} + 1$ IS DIVISIBLE BY 7 FOR ANY NON-NEGATIVE INTEGER n.**

$2^3 \equiv 1 \bmod 7 \Rightarrow 2^{3n+1} = (2^3)^n \cdot 2 \equiv 1^n \cdot 2 \bmod 7 \equiv 2 \bmod 7$

$$2^{3n+1} \equiv 2 \bmod 7$$

$4^3 = 2^3 \cdot 2^3 \equiv 1 \cdot 1 \bmod 7 \equiv 1 \bmod 7 \Rightarrow 4^{3n+1} = (4^3)^n \cdot 4 \equiv 4 \bmod 7$

$$4^{3n+1} \equiv 4 \bmod 7$$

$$\begin{aligned} 4^{3n+1} + 2^{3n+1} + 1 &\equiv (4+2+1) \bmod 7 \\ &\equiv 7 \bmod 7 \\ &\equiv 0 \bmod 7 \end{aligned}$$

SO $7 \mid 4^{3n+1} + 2^{3n+1} + 1$.

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2. COMPUTE $\displaystyle\lim_{n \to \infty} \frac{2 \ln 2 + 3 \ln 3 + \cdots + n \ln n}{n^2 \ln n}$

CONSIDER THE FUNCTION $f(x) = x \ln x$. THIS IS POSITIVE AND INCREASING ON $x > 1$ SO

$$\int_1^n t\ln t\,dt \le \sum_{k=2}^{n} k\ln k \le \int_2^{n+1} t\ln t\,dt.$$

SINCE

$$\int t\ln t\,dt = \frac{t^2}{2}\ln t - \frac{t^2}{4} + C,$$

$$\int_1^n t\ln t\,dt = \frac{n^2}{2}\ln n - \frac{n^2}{4} + \frac{1}{4} \quad \text{AND}$$

$$\int_2^{n+1} t\ln t\,dt = \frac{(n+1)^2}{2}\ln(n+1) - \frac{(n+1)^2}{4} - (2\ln 2 - 1) \quad \text{SO}$$

$$(*) \qquad \frac{n^2}{2}\ln n - \frac{n^2}{4} \le \sum_{k=2}^{n} k\ln k \le \frac{(n+1)^2}{2}\ln(n+1) - \frac{(n+1)^2}{4} - 2\ln 2 + 1$$

(WE DROPPED THE $\frac{1}{4}$ IN THE FIRST TERM). NOW,

$$\lim_{n\to\infty} \frac{\frac{n^2}{2}\ln n - \frac{n^2}{4}}{n^2\ln n} = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{4\ln n}\right) = \frac{1}{2}$$

AND

$$\lim_{n\to\infty} \frac{\frac{(n+1)^2}{2}\ln(n+1) - \frac{(n+1)^2}{4} - 2\ln 2 + 1}{n^2\ln n} =$$

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{2n^2}\,\frac{\ln(n+1)}{\ln n} - \frac{(n+1)^2}{4n^2\ln n} - \frac{2\ln 2 - 1}{n^2\ln n}\right) = \frac{1}{2} - 0 - 0 = \frac{1}{2}.$$

THUS, BY THE SQUEEZE THEOREM, (*) GIVES

$$\lim_{n\to\infty} \frac{\sum_{k=2}^{n} k\ln k}{n^2\ln n} = \frac{1}{2}.$$

3. **DETERMINE WHETHER THE FOLLOWING MATRIX IS SINGULAR OR NON-SINGULAR:**

$$\begin{bmatrix} 54401 & 57668 & 15982 & 103790 \\ 33223 & 26563 & 23165 & 71489 \\ 36799 & 37189 & 16596 & 46152 \\ 21689 & 55538 & 79922 & 51237 \end{bmatrix}$$

WE SHOW THAT THE DETERMINANT IS $\equiv 1 \bmod 2$ AND THEREFORE CANNOT BE 0 SO THE MATRIX IS NONSINGULAR.

THE DETERMINANT IS A POLYNOMIAL IN THE ENTRIES SO BASIC PROPERTIES OF CONGRUENCES IMPLY THAT ITS VALUE MOD 2 CAN BE COMPUTED BY REPLACING EACH ENTRY BY ITS LEAST RESIDUE MOD 2. IN THIS CASE THAT GIVES

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= -1 \equiv 1 \bmod 2.$$

4. FOR A PARTITION π OF $\{1,2,3,4,5,6,7,8,9\}$ LET $\pi(x)$ BE THE NUMBER OF ELEMENTS IN THE PART CONTAINING x. PROVE THAT FOR ANY TWO PARTITIONS π AND π', THERE ARE TWO DISTINCT NUMBERS x AND y IN $\{1,2,3,4,5,6,7,8,9\}$ SUCH THAT $\pi(x) = \pi(y)$ AND $\pi'(x) = \pi'(y)$.

[A PARTITION OF A SET S IS A COLLECTION OF DISJOINT SUBSETS (PARTS) WHOSE UNION IS S.]

SUPPOSE NOT. THEN THE ORDERED PAIRS

$$(\pi(1), \pi'(1)),\ (\pi(2), \pi'(2)), \ldots, (\pi(9), \pi'(9))$$

ARE ALL DISTINCT (SO THERE ARE 9 OF THEM). $\pi(x)$ AND $\pi'(x)$ CANNOT ASSUME MORE THAN THREE DISTINCT VALUES (FOR THEN THE NUMBER OF ELEMENTS IN $\{1,2,3,4,5,6,7,8,9\}$

WOULD BE AT LEAST $1+2+3+4=10$). SINCE THERE ARE 9 DISTINCT ORDERED PAIRS ABOVE, EVERY VALUE ASSUMED BY π MUST OCCUR AT LEAST 3 TIMES IN THE LIST AS A FIRST COORDINATE. SIMILARLY, EACH VALUE ASSUMED BY π' MUST OCCUR AT LEAST 3 TIMES AS A SECOND COORDINATE. IT FOLLOWS THAT IT CANNOT BE THE CASE THAT ALL OF THE VALUES OF $\pi(x)$ AND ALL OF THE VALUES OF $\pi'(x)$ ARE ≤ 3. THE REASON IS AS FOLLOWS. EACH VALUE $\pi(x)$ MUST OCCUR THREE TIMES SO $(\pi(x), 1)$, $(\pi(x), 2)$ AND $(\pi(x), 3)$ MUST ALL OCCUR. THE ONLY WAY TO OBTAIN 9 DISTINCT PAIRS IS FOR $\pi(x)$ TO TAKE ALL OF THE VALUES 1, 2 AND 3. THUS,

THE SET OF ORDERED PAIRS ABOVE IS PRECISELY

$$(1,1),\ (1,2),\ (1,3),\ (2,1),\ (2,2),\ (2,3),\ (3,1),\ (3,2),\ (3,3)$$

BUT THEN $\pi(x) = 2$ OCCURS 3 TIMES SO THERE ARE PRECISELY 3 x S IN $\{1, \ldots, 9\}$ FOR WHICH $\pi(x) = 2$. BUT THIS IS CLEARLY IMPOSSIBLE SINCE THE x S FOR WHICH $\pi(x) = 2$ OCCUR IN PAIRS SO THE NUMBER OF SUCH x IS EVEN.

THUS, THERE MUST BE AN x FOR WHICH EITHER $\pi(x) \geq 4$ OR $\pi'(x) \geq 4$. IF $\pi(x) \geq 4$ WE CAN SELECT DISTINCT NUMBERS x_1, x_2, x_3, x_4 WITH $\pi(x_1) = \pi(x_2) = \pi(x_3) = \pi(x_4)$. SINCE π' CAN ASSUME AT MOST 3 VALUES, $\pi'(x_1)$, $\pi'(x_2)$, $\pi'(x_3)$, $\pi'(x_4)$ CANNOT ALL BE DISTINCT SO, FOR SOME $i, j = 1, 2, 3, 4$, $i \neq j$, $\pi'(x_i) = \pi'(x_j)$. THUS,

$$(\pi(x_i), \pi'(x_i)) = (\pi(x_j), \pi'(x_j))$$

AND THIS IS A CONTRADICTION. THE ARGUMENT IS THE SAME IF $\pi'(x) \geq 4$.

5. A LATTICE POINT IN THE PLANE IS A POINT WHOSE COORDINATES ARE BOTH INTEGERS. FIND ALL THE LATTICE POINTS ON THE HYPERBOLA

$$x^2 - 2y^2 = 3$$

WE ARE ASKED TO FIND ALL INTEGER SOLUTIONS TO THE EQUATION $x^2 - 2y^2 = 3$. BUT THIS IMPLIES

$$x^2 - 2y^2 \equiv 0 \bmod 3$$

AND, SINCE $-2 \equiv 1 \bmod 3$,

$$x^2 + y^2 \equiv 0 \bmod 3$$

THE QUADRATIC RESIDUES MOD 3 ARE 0 AND 1 AND THE ONLY SUM OF TWO THESE THAT IS $\equiv 0 \bmod 3$ IS $0+0$. THUS, $x^2 \equiv 0 \bmod 3$ AND $y^2 \equiv 0 \bmod 3$ SO $3 \mid x^2$ AND $3 \mid y^2$. BUT 3 IS PRIME SO $3 \mid x$ AND $3 \mid y$. THUS, $x = 3k_1$ AND $y = 3k_2$ FOR SOME INTEGERS k_1 AND k_2. BUT THEN

$$x^2 - 2y^2 = 9k_1^2 - 18k_2^2 = 9(k_1^2 - 2k_2^2)$$

AND THIS CANNOT BE 3 UNLESS $k_1^2 - 2k_2^2 = \frac{1}{3}$ WHICH IS IMPOSSIBLE SINCE k_1 AND k_2 ARE INTEGERS.

THERE ARE NO LATTICE POINTS ON THE HYPERBOLA $x^2 - 2y^2 = 3$.