PROBLEM SET 4 (SOLUTIONS)

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1. LET a, b, AND c BE ODD INTEGERS. PROVE THAT THE QUADRATIC POLYNOMIAL $ax^2 + bx + c$ CANNOT HAVE A RATIONAL ROOT.

THE POLYNOMIAL $ax^2 + bx + c$ HAS TWO (GENERALLY, COMPLEX) ROOTS m AND n AND

$$mn = \frac{c}{a}$$

($a \neq 0$ SINCE IT IS AN ODD INTEGER AND $x^2 + \frac{b}{a}x + \frac{c}{a}$ FACTORS AS $(x-m)(x-n) = x^2 - (m+n)x + mn$ SO $\frac{b}{a} = -(m+n)$ AND $\frac{c}{a} = mn$).

SINCE a AND c ARE INTEGERS, $\frac{c}{a}$ IS RATIONAL SO IF ONE OF THE ROOTS IS RATIONAL, SO IS THE OTHER.

THUS, IF WE NOW ASSUME THAT ax^2+bx+c HAS A RATIONAL ROOTS WE CAN CONCLUDE THAT BOTH m AND n ARE RATIONAL. CONSEQUENTLY, WE CAN FACTOR

$$ax^2 + bx + c = (px+q)(rx+s)$$

WHERE p, q, r, AND s ARE INTEGERS. SINCE

$$a = pr$$

IS ODD, BOTH p AND r MUST BE ODD. SINCE

$$c = qs$$

IS ODD, BOTH q AND s MUST BE ODD. SINCE p, q, r, AND s ARE

ALL ODD, $ps + qr$ IS EVEN. BUT

$$b = ps + qr$$

SO THIS CONTRADICTS THE ASSUMPTION THAT a, b, AND c ARE ALL ODD. THUS, $ax^2 + bx + c$ CAN HAVE NO RATIONAL ROOT.

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2. EVALUATE THE SUM $\sum_{i=1}^{n} i \binom{n}{i}$ FOR ANY INTEGER $n \geq 1$.

$$\sum_{i=1}^{n} i \binom{n}{i} = \sum_{i=1}^{n} i \left(\frac{n}{i}\right) \binom{n-1}{i-1} = n \sum_{i=1}^{n} \binom{n-1}{i-1}$$

$$= n \sum_{j=0}^{n-1} \binom{n-1}{j}.$$

BUT

$$(1+x)^{n-1} = \sum_{j=0}^{n-1} \binom{n-1}{j} x^j$$

SO, WHEN $x = 1$,

$$\sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1},$$

THUS,

$$\sum_{i=1}^{n} i \binom{n}{i} = n\, 2^{n-1}.$$

<u>ALTERNATIVE</u>: DIFFERENTIATE $(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$ AND EVALUATE AT $x = 1$.

3, SHOW THAT FOR ANY REAL NUMBERS a, b, AND c THE EQUATION

$$4ax^3 + 3bx^2 + 2cx = a+b+c$$

HAS AT LEAST ONE ROOT BETWEEN 0 AND 1.

<u>NOTE</u> : APPLYING THE INTERMEDIATE VALUE THEOREM TO

$$f(x) = 4ax^3 + 3bx^2 + 2cx - (a+b+c)$$

DOESN'T WORK SO WELL SINCE

$$f(0) = -(a+b+c) \qquad f(1) = 3a+2b+c$$

DO NOT OBVIOUSLY HAVE OPPOSITE SIGN.

INSTEAD, APPLY ROLLE'S THEOREM TO

$$F(x) = ax^4 + bx^3 + cx^2 - (a+b+c)x$$

(SO $F'(x) = f(x)$).

$$F(0) = 0 = F(1)$$

SO $\exists\ x_0 \in (0,1)$ S.T. $F'(x_0) = 0$, I.E.,

$$f(x_0) = 0.$$

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4. LET A BE A LINEAR TRANSFORMATION ON AN n-DIMENSIONAL VECTOR SPACE V AND SUPPOSE A HAS A SET OF $n+1$ EIGENVECTORS, ANY n OF WHICH ARE LINEARLY INDEPENDENT. PROVE THAT A IS A MULTIPLE OF THE IDENTITY.

LET $v_1, \ldots, v_{n+1}$ BE THE EIGENVECTORS AND $\lambda_1, \ldots, \lambda_{n+1}$ THE CORRESPONDING EIGENVALUES. FOR EACH $i = 1, \ldots, n+1$

$$B_i = \{ v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n+1} \}$$

IS A BASIS FOR V. WITH RESPECT TO B_i THE MATRIX OF A IS

$$\mathrm{diag}\,(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{n+1})$$

SO

$$\mathrm{Tr}\, A = S - \lambda_i$$

WHERE

$$S = \lambda_1 + \cdots + \lambda_{n+1}.$$

THIS IS TRUE $\forall\ i = 1, \ldots, n+1$ SO IF $1 \le i < j \le n+1$,

$$S - \lambda_i = S - \lambda_j$$

$$\lambda_i = \lambda_j .$$

ALL OF THE EIGENVALUES ARE EQUAL (TO, SAY, λ) SO

$$A = \lambda I .$$

5. DEFINE A SEQUENCE BY $a_0 = 1$, TOGETHER WITH THE RULES $a_{2n+1} = a_n$ AND $a_{2n+2} = a_n + a_{n+1}$ FOR EACH $n \geq 0$. PROVE THAT EVERY POSITIVE RATIONAL NUMBER APPEARS IN THE SET

$$\left\{ \frac{a_{n-1}}{a_n} : n \geq 1 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \dots \right\}.$$

DENOTE THIS SET BY S. WE SHOW THAT FOR ALL POSITIVE INTEGERS a AND b, $\frac{a}{b}$ IS IN S. THE PROOF IS BY INDUCTION ON $a+b$. IF $a+b = 2$ (THE SMALLEST VALUE POSSIBLE), THEN $a=1$ AND $b=1$ SO $\frac{a}{b} = \frac{a_0}{a_1}$ IS IN S. NOW SUPPOSE THAT $\frac{a}{b}$ IS IN S WHENEVER a AND b ARE POSITIVE INTEGERS WITH $a+b < m$. ASSUME A AND B ARE POSITIVE INTEGERS WITH $A+B = m$ AND CONSIDER $\frac{A}{B}$.

IF $A = B$, THEN $\frac{A}{B} = 1 = \frac{a_0}{a_1}$ IS IN S. IF $A > B$, THEN $(A-B)+B$ IS LESS THAN $A+B = m$ SO THE INDUCTION HYPOTHESIS IMPLIES THAT $\frac{A-B}{B}$ IS IN S, I.E., $\frac{A-B}{B} = \frac{a_{n-1}}{a_n}$ FOR SOME $n \geq 1$. BUT THEN

$$\frac{A-B}{B} = \frac{A}{B} - 1 = \frac{A}{B} - \frac{a_n}{a_n} = \frac{a_{n-1}}{a_n}$$

SO

$$\frac{A}{B} = \frac{a_{n-1} + a_n}{a_n} = \frac{a_{2n}}{a_{2n+1}}$$

IS IN S. IF $A < B$, THEN $A + (B-A)$ IS LESS THAN $A+B = m$ SO $\frac{A}{B-A}$ IS IN S, I.E., $\frac{A}{B-A} = \frac{a_{n-1}}{a_n}$ FOR SOME $n \geq 1$. THEN $\frac{B-A}{A} = \frac{a_n}{a_{n-1}}$ SO $\frac{B}{A} = \frac{a_n + a_{n-1}}{a_{n-1}}$ AND $\frac{A}{B} = \frac{a_{n-1}}{a_n + a_{n-1}} = \frac{a_{2n-1}}{a_{2n}}$ IS IN S.