PROBLEM SOLVING SEMINAR

PROBLEM SET 5 (SOLUTIONS)

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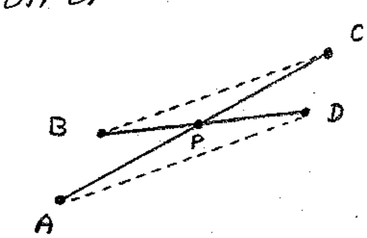
1. THERE ARE $2n$ BALLS IN THE PLANE. NO TWO BALLS TOUCH EACH OTHER AND NO THREE BALLS ARE ON THE SAME LINE. n BALLS ARE RED AND n BALLS ARE BLUE. SHOW THAT THERE IS AT LEAST ONE WAY TO DRAW n LINE SEGMENTS BY CONNECTING EACH BALL TO A DIFFERENT COLORED BALL SO THAT NO TWO LINE SEGMENTS INTERSECT.

CHOOSE n LINE SEGMENTS CONNECTING A RED BALL WITH A BLUE BALL SO THAT THE SUM OF THEIR LENGTHS IS AS SMALL AS POSSIBLE (THERE ARE ONLY FINITELY MANY CHOICES OF n SEGMENTS CONNECTING A RED BALL WITH A BLUE BALL SO AT LEAST ONE CHOICE WILL HAVE A MINIMAL SUCH SUM). WE CLAIM THAT NO TWO LINE SEGMENTS IN THIS CHOICE CAN INTERSECT. INDEED, SUPPOSE A, B ARE RED BALLS AND C, D ARE BLUE BALLS AND THE SEGMENTS AC AND BD INTERSECT AT SOME POINT P. SINCE THE LENGTH OF ANY SIDE OF A TRIANGLE IS LESS THAN THE SUM OF THE LENGTHS OF THE OTHER TWO SIDES WE HAVE



$$AD < AP + PD \quad \text{AND} \quad BC < BP + PC$$

SO

$$AD + BC < AP + PC + BP + PD = AC + BD$$

THUS, REPLACING AC BY AD AND BD BY BC WE OBTAIN LINE SEGMENTS JOINING RED BALLS TO BLUE BALLS WITH A SMALLER SUM AND THIS IS A CONTRADICTION.

2. PROVE THAT NO EQUILATERAL TRIANGLE IN THE PLANE CAN HAVE ALL OF ITS VERTICES AT LATTICE POINTS (I.E., AT POINTS WITH BOTH COORDINATES INTEGERS).

SUPPOSE THERE IS SUCH A TRIANGLE. THEN THERE MUST BE SUCH A TRIANGLE WITH ONE VERTEX AT THE ORIGIN (TWO INTEGER TRANSLATIONS, ONE HORIZONTAL AND ONE VERTICAL, WILL MOVE A VERTEX TO $(0,0)$ AND THE OTHER TWO VERTICES TO NEW LATTICE POINTS). SUPPOSE THE VERTICES ARE $(0,0)$, (a,b), AND (c,d). WE CAN ASSUME THAT THE GREATEST COMMON DIVISOR OF a, b, c, AND d IS 1 SINCE OTHERWISE WE COULD DIVIDE ALL OF THE COORDINATES BY IT, YIELDING $(0,0)$ AND TWO NEW LATTICE POINTS THAT ARE STILL THE VERTICES OF AN EQUILATERAL TRIANGLE. NOW WE HAVE

$$a^2+b^2 = c^2+d^2 = (c-a)^2+(d-b)^2.$$

THE SECOND EQUALITY GIVES $a^2+b^2 = 2(ac+bd)$ SO $c^2+d^2 = 2(ac+bd)$. THUS,

$$a^2+b^2+c^2+d^2 = 4(ac+bd) \equiv 0 \bmod 4.$$

NOW, EVERY SQUARE IS $\equiv \bmod 4$ TO 0 OR 1, BUT $a, b, c,$ AND d ARE NOT ALL EVEN SO a^2, b^2, c^2 AND d^2 MUST ALL BE $\equiv 1 \bmod 4$. IN PARTICULAR, THEY ARE ALL ODD SO $a, b, c,$ AND d ARE ALL ODD. THUS, $c-a$ AND $d-b$ ARE

EVEN SO

$$(c-a)^2 + (d-b)^2 \equiv 0 \bmod 4.$$

THIS GIVES

$$a^2 + b^2 \equiv 0 \bmod 4$$

WHEREAS, IN FACT,

$$a^2 + b^2 \equiv (1+1) \bmod 4 \equiv 2 \bmod 4$$

AND THIS IS A CONTRADICTION.

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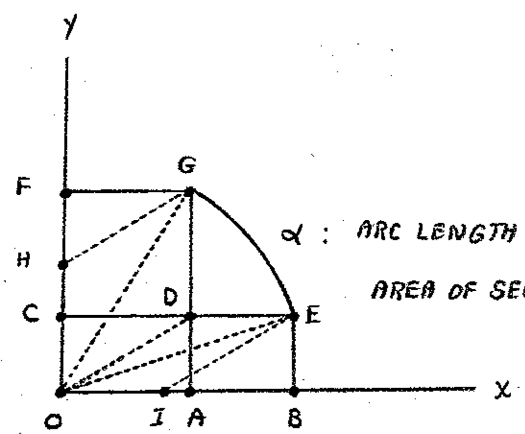
3. LET α BE ANY ARC OF THE UNIT CIRCLE LYING ENTIRELY IN THE FIRST QUADRANT. LET A_1 BE THE AREA OF THE REGION BELOW α AND ABOVE THE X-AXIS AND LET A_2 BE THE AREA OF THE REGION TO THE LEFT OF α AND TO THE RIGHT OF THE Y-AXIS. SHOW THAT $A_1 + A_2$ DEPENDS ONLY ON THE LENGTH OF α AND NOT ON ITS POSITION.



α : ARC LENGTH $= S$

AREA OF SECTOR OEG $= \frac{1}{2} r^2 S = \frac{1}{2} S$

$= |OEG|$

$$
\begin{aligned}
A_1 + A_2 &= |ABED| + |DEG| + |CDGF| + |DEG| \\
&= |ABED| + |CDGF| + 2\,|DEG| \\
&= |OIED| + |ODGH| + 2\,|DEG| \\
&= 2\,|OED| + 2\,|ODG| + 2\,|DEG| \\
&= 2\,(\,|OED| + |ODG| + |DEG|\,) \\
&= 2\,|OEG| \\
&= 2\left(\tfrac{1}{2}S\right) \\
&= S
\end{aligned}
$$

SO, IN FACT, $A_1 + A_2$ IS PRECISELY THE LENGH OF α.

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4. PROVE THAT $\sin^2 x < \sin(x^2)$ FOR $0 < x \leq \sqrt{\frac{\pi}{2}}$.

FIRST CONSIDER THE INTERVAL $1 \leq x \leq \sqrt{\frac{\pi}{2}}$. THEN $1 \leq x \leq x^2 \leq \frac{\pi}{2}$. MOREOVER, THE SINE FUNCTION IS INCREASING ON THIS INTERVAL SO $\sin x \leq \sin(x^2)$. BUT ALSO, $0 < \sin x < 1$ SO $\sin^2 x < \sin x$ AND THEREFORE

$$\sin^2 x < \sin(x^2)$$

ON $1 \leq x \leq \sqrt{\frac{\pi}{2}}$.

NOW CONSIDER THE INTERVAL $0 < x < 1$. THEN $0 < x^2 < x < 1$. ON THIS INTERVAL THE COSINE FUNCTION IS DECREASING SO

$$0 < \cos x < \cos(x^2).$$

SINCE $0 < \sin x$ ON $0 < x < 1$,

$$0 < \sin x \cos x < \sin x \cos(x^2).$$

MOREOVER, $\sin x < x$ ON $0 < x < 1$ BECAUSE $(\sin x - x)' = \cos x - 1 < 0$ ON $0 < x < 1$. THUS,

$$0 < \sin x \cos x < x \cos(x^2)$$

ON $0 < x < 1$. NOW INTEGRATE TO OBTAIN

$$\int_0^x \sin t \cos t \, dt < \int_0^x t \cos(t^2) dt$$

$$\frac{1}{2} \sin^2 t \Big|_0^x < \frac{1}{2} \sin(t^2) \Big|_0^x$$

$$\sin^2 x < \sin(x^2)$$

FOR $0 < x < 1$. THUS, $\sin^2 x < \sin(x^2)$ EVERYWHERE ON $0 < x \le \sqrt{\frac{\pi}{2}}$.

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5. LET A BE A POSITIVE REAL NUMBER. CONSIDER THE SET OF ALL SEQUENCES $x_0, x_1, x_2, \ldots$ OF POSITIVE REAL NUMBERS FOR WHICH $\sum_{n=0}^{\infty} x_n = A$. FIND ALL OF THE POSSIBLE VALUES OF $\sum_{n=0}^{\infty} x_n^2$.

NOTE THAT $\sum_{n=0}^{\infty} x_n^2$ MUST CONVERGE BECAUSE ITS PARTIAL SUMS ARE INCREASING AND BOUNDED ABOVE.

$$0 < \sum_{n=0}^{k} x_n^2 = \left(\sum_{n=0}^{k} x_n\right)^2 - \sum_{0 \le n < m \le k} 2x_n x_m < \left(\sum_{n=0}^{k} x_n\right)^2 - 2x_0 x_1$$

$$< \left(\sum_{n=0}^{\infty} x_n\right)^2 - 2x_0 x_1 = A^2 - 2x_0 x_1$$

THUS, $\sum_{n=0}^{\infty} x_n^2 \le A^2 - 2x_0 x_1 < A^2$

SO THE ONLY POSSIBLE VALUES OF $\sum_{n=0}^{\infty} x_n^2$ ARE IN THE INTERVAL $(0, A^2)$. WE NOW SHOW THAT, IN FACT, EVERY S IN $(0, A^2)$ IS $\sum_{n=0}^{\infty} x_n^2$ FOR SOME $x_0, x_1, \ldots$ WITH $\sum_{n=0}^{\infty} x_n = A$.

LET $0 < S < A^2$. NOW NOTE THAT, FOR ANY $0 < r < 1$, $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$

AND THIS IMPLIES THAT

$$\sum_{n=0}^{\infty} A(1-r)r^n = A(1-r)\sum_{n=0}^{\infty} r^n = A.$$

THUS, FOR ANY $0 < r < 1$, IF $x_n = A(1-r)r^n$, THEN $\sum_{n=0}^{\infty} x_n = A$. WE SHOW THAT r CAN BE CHOSEN SO THAT $\sum_{n=0}^{\infty} x_n^2 = S$. NOTE THAT

$$\sum_{n=0}^{\infty} x_n^2 = \sum_{n=0}^{\infty} (A(1-r)r^n)^2 = A^2(1-r)^2 \sum_{n=0}^{\infty} (r^2)^n$$

$$= A^2(1-r)^2 \frac{1}{1-r^2} = A^2\left(\frac{1-r}{1+r}\right).$$

SINCE $0 < \frac{S}{A^2} < 1$ AND $0 < \frac{1-r}{1+r} < 1$ WE MAY SET $\frac{1-r}{1+r} = \frac{S}{A^2}$ AND SOLVE FOR r TO OBTAIN $r = \dfrac{1 - \frac{S}{A^2}}{1 + \frac{S}{A^2}}$.

FOR THIS r (WHICH IS IN $(0,1)$), $\sum_{n=0}^{\infty} x_n^2 = S$.