PROBLEM SOLVING SEMINAR

# PROBLEM SET 5

1. THERE ARE $2n$ BALLS IN THE PLANE. NO TWO BALLS TOUCH EACH OTHER AND NO THREE BALLS ARE ON THE SAME LINE. $n$ BALLS ARE RED AND $n$ BALLS ARE BLUE. SHOW THAT THERE IS AT LEAST ONE WAY TO DRAW $n$ LINE SEGMENTS BY CONNECTING EACH BALL TO A DIFFERENT COLORED BALL SO THAT NO TWO LINE SEGMENTS INTERSECT.

2. PROVE THAT NO EQUILATERAL TRIANGLE IN THE PLANE CAN HAVE ALL OF ITS VERTICES AT LATTICE POINTS (I.E., AT POINTS WITH BOTH COORDINATES INTEGERS).

3. LET $\alpha$ BE ANY ARC OF THE UNIT CIRCLE LYING ENTIRELY IN THE FIRST QUADRANT. LET $A_1$ BE THE AREA OF THE REGION BELOW $\alpha$ AND ABOVE THE X-AXIS AND LET $A_2$ BE THE AREA OF THE REGION TO THE LEFT OF $\alpha$ AND TO THE RIGHT OF THE y-AXIS. SHOW THAT $A_1 + A_2$ DEPENDS ONLY ON THE LENGTH OF $\alpha$ AND NOT ON ITS POSITION.

4. PROVE THAT $\sin^2 x < \sin(x^2)$ FOR $0 < x \le \sqrt{\frac{\pi}{2}}$.

5. LET $A$ BE A POSITIVE REAL NUMBER. CONSIDER THE SET OF ALL SEQUENCES $x_0, x_1, x_2, \ldots$ OF POSITIVE REAL NUMBERS FOR WHICH $\sum_{n=0}^{\infty} x_n = A$. FIND ALL OF THE POSSIBLE VALUES OF $\sum_{n=0}^{\infty} x_n^2$.