PROBLEM SOLVING SEMINAR

PROBLEM SET 6 (SOLUTIONS)

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1. SUM THE SERIES

$$\frac{3}{1\times2\times3} + \frac{5}{2\times3\times4} + \frac{7}{3\times4\times5} + \frac{9}{4\times5\times6} + \cdots$$

THE SERIES IS $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)(n+2)}$

PARTIAL FRACTIONS:

$$\frac{2n+1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}$$

$$2n+1 = A(n+1)(n+2) + Bn(n+2) + Cn(n+1)$$

$n=0 \rightarrow A = \frac{1}{2}$

$n=-1 \rightarrow B = 1$

$n=-2 \rightarrow C = -\frac{3}{2}$

SO THE SERIES IS $\displaystyle\sum_{n=1}^{\infty} \left[\frac{\frac{1}{2}}{n} + \frac{1}{n+1} - \frac{\frac{3}{2}}{n+2} \right]$

THE k^{TH} PARTIAL SUM IS

$$S_k = \frac{1}{2}\sum_{n=1}^{k}\left(\frac{1}{n} + \frac{2}{n+1} - \frac{3}{n+2}\right) =$$

$$\frac{1}{2}\Big[\left(\frac{1}{1}+\frac{2}{2}-\frac{3}{3}\right)+\left(\frac{1}{2}+\frac{2}{3}-\frac{3}{4}\right)+\left(\frac{1}{3}+\frac{2}{4}-\frac{3}{5}\right)+$$

$$\left(\frac{1}{4}+\frac{2}{5}-\frac{3}{6}\right)+\left(\frac{1}{5}+\frac{2}{6}-\frac{3}{7}\right)+\cdots+$$

$$\left(\frac{1}{k-2}+\frac{2}{k-1}-\frac{3}{k}\right)+\left(\frac{1}{k-1}+\frac{2}{k}-\frac{3}{k+1}\right)+\left(\frac{1}{k}+\frac{2}{k+1}-\frac{3}{k+2}\right)\Big]$$

$$= \frac{1}{2}\left[\frac{1}{1}+\frac{2}{2}+\frac{1}{2}-\frac{3}{k+1}+\frac{2}{k+1}-\frac{3}{k+2}\right]$$

$$= \frac{1}{2}\left[\frac{5}{2}-\frac{1}{k+1}-\frac{3}{k+2}\right] \longrightarrow \frac{5}{4} \text{ AS } k\to\infty$$

SO

$$\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)(n+2)} = \frac{5}{4}$$

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2. **Given an infinite set of points in the plane, prove that if all of the distances between them are integers, then the points must lie on a single straight line.**

Suppose the given set contains three noncollinear points A, B and C. Denote the lengths of the segments AB and AC by $|AB| = r$ and $|AC| = s$ so that r and s are integers. Suppose P is any point an integral distance from both A and B. The triangle inequality gives $|PA| \le |PB| + r$ and $|PB| \le |PA| + r$ so $|PA| - |PB| \le r$ and $|PB| - |PA| \le r$, i.e.

$$\big|\,|PA| - |PB|\,\big| \le r.$$

Since r is an integer,

$$\big|\,|PA| - |PB|\,\big| = 0, 1, \ldots, r-1, r.$$

Thus, P must lie in one of the sets

$$H_j = \{X: \big|\,|XA| - |XB|\,\big| = j\}, \qquad j = 0, 1, \ldots, r-1, r.$$

If $j = 0$, $|XA| = |XB|$ so H_0 is the perpendicular bisector of the segment AB. If $j = r$, then $|XA| - |XB| = |AB|$, or $|XB| - |XA| = |AB|$ so $|AB| + |BX| = |AX|$, or $|XA| + |AB| = |XB|$ and so X must lie on the line H_r joining A and B and be outside the segment AB itself. If $j = 1, \ldots, r-1$, then H_j is the set of points the difference of whose distances from A and B (larger minus smaller) is j (which is less than the distance from A to B) and this is the definition of the hyperbola with foci at A and B. Thus, any such P must lie on the perpendicular bisector of AB, the line containing A and B (outside AB) or one of these hyperbolas. In the same way, P is an integral distance from both A and C so it must lie on one of the sets

$$K_i = \{X : \mid |XA| - |XC| \mid = i\}, \qquad i = 0, 1, \ldots, s-1, s,$$

where K_0 is the perpendicular bisector of the segment AC, K_s is the part of the line joining A and C outside the segment AC and K_i, $i = 1, \ldots, s-1$, is a hyperbola with foci at A and C.

Now, any point in the given set must be in one of the sets $H_j \cap K_i$. Since A, B and C are noncollinear, the lines through A and B and through A and C are distinct so no H_j can coincide with any K_i. Thus, for all j and i, $H_j \cap K_i$ is the intersection of two distinct curves of degree at most 2 so it contains at most 4 points. Consequently, our set can contain at most $4(r+1)(s+1)$ points and this contradicts the fact that it was assumed infinite. All the points must therefore be collinear.

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3. CAN A COUNTABLY INFINITE SET HAVE AN UNCOUNTABLE FAMILY OF INFINITE SUBSETS SUCH THAT THE INTERSECTION OF ANY TWO OF THEM IS FINITE. IF NO, PROVE IT ; IF YES, CONSTRUCT AN EXAMPLE.

THE ANSWER IS "YES". HERE'S AN EXAMPLE. THE SET $\mathbb{Q}$ OF RATIONAL NUMBERS IS COUNTABLY INFINITE. FOR EVERY REAL NUMBER α CHOOSE A SEQUENCE OF DISTINCT RATIONALS CONVERGING TO α AND LET S_α BE THE TERMS OF THE SEQUENCE. IF $\alpha \neq \beta$, THEN $S_\alpha \cap S_\beta$ MUST BE FINITE SINCE OTHERWISE THE TWO SEQUENCES WOULD HAVE A COMMON SUBSEQUENCE AND THIS WOULD HAVE TO CONVERGE TO BOTH α AND β, WHICH IS IMPOSSIBLE. IN PARTICULAR, $S_\alpha \neq S_\beta$ SO $\{S_\alpha\}_{\alpha \in \mathbb{R}}$ IS AN UNCOUNTABLE FAMILY OF INFINITE SUBSETS OF $\mathbb{Q}$ WITH THE REQUIRED PROPERTY.

4. CONSIDER THE POWER SERIES EXPANSION $\dfrac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n$.

PROVE THAT FOR EACH INTEGER $n \geq 0$ THERE IS AN INTEGER $m \geq 0$ SUCH THAT

$$a_m = a_n^2 + a_{n+1}^2 .$$

WE WILL SIMPLY FIND THE COEFFICIENTS a_n AND COMPUTE $a_n^2 + a_{n+1}^2$. FIRST FACTOR $1-2x-x^2$ BY SOLVING

$$1-2x-x^2 = 0$$

QUADRATIC FORMULA GIVES

$$x = 1 \pm \sqrt{2}$$

SO

$$1-2x-x^2 = (1-(1+\sqrt{2})x)(1-(1-\sqrt{2})x)$$

NOW A PARTIAL FRACTIONS DECOMPOSITION GIVES

$$\frac{1}{1-2x-x^2} = \frac{1}{(1-(1+\sqrt{2})x)(1-(1-\sqrt{2})x)} = \frac{1}{2\sqrt{2}}\left(\frac{1+\sqrt{2}}{1-(1+\sqrt{2})x} - \frac{1-\sqrt{2}}{1-(1-\sqrt{2})x}\right)$$

AND THE GEOMETRIC SERIES GIVES, FOR $|x| < \frac{1}{1+\sqrt{2}}$,

$$\frac{1}{1-2x-x^2} = \frac{1}{2\sqrt{2}} \sum_{n=0}^{\infty} \left((1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}\right) x^n$$

SO $a_n = \frac{1}{2\sqrt{2}}\left((1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}\right)$. THUS,

$$a_n^2 = \frac{1}{8}\left[(1+\sqrt{2})^{2n+2} - 2(1+\sqrt{2})^{n+1}(1-\sqrt{2})^{n+1} + (1-\sqrt{2})^{2n+2}\right]$$

$$= \frac{1}{8}\left[(1+\sqrt{2})^{2n+2} - 2(-1)^{n+1} + (1-\sqrt{2})^{2n+2}\right]$$

AND

$$a_{n+1}^2 = \frac{1}{8}\left[(1+\sqrt{2})^{2n+4} - 2(-1)^{n+2} + (1-\sqrt{2})^{2n+4}\right]$$

ADDING THESE GIVES

$$a_n^2 + a_{n+1}^2 = \frac{1}{8}\left[(1+\sqrt{2})^{2n+2} + (1+\sqrt{2})^{2n+4} + (1-\sqrt{2})^{2n+2} + (1-\sqrt{2})^{2n+4}\right]$$

$$= \frac{1}{8}\left[(1+\sqrt{2})^{2n+2}\underbrace{(1+(1+\sqrt{2})^2)}_{4+2\sqrt{2}} + (1-\sqrt{2})^{2n+2}\underbrace{(1+(1-\sqrt{2})^2)}_{4-2\sqrt{2}}\right]$$

$$a_n^2 + a_{n+1}^2 = \frac{1}{8}\left[(1+\sqrt{2})^{2n+2}2(2+\sqrt{2}) + (1-\sqrt{2})^{2n+2}2(2-\sqrt{2})\right]$$

$$= \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{2n+2}\frac{2+\sqrt{2}}{\sqrt{2}} + (1-\sqrt{2})^{2n+2}\frac{2-\sqrt{2}}{\sqrt{2}}\right]$$

$$= \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{2n+2}(1+\sqrt{2}) - (1-\sqrt{2})^{2n+2}(1-\sqrt{2})\right]$$

$$= \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{2n+3} - (1-\sqrt{2})^{2n+3}\right]$$

$$= a_{2n+2}$$

SO $m = 2n+2$.

5. LET G BE A GROUP WITH IDENTITY e AND $\phi : G \to G$ A FUNCTION SUCH THAT

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

WHENEVER $g_1 g_2 g_3 = e = h_1 h_2 h_3$. PROVE THAT THERE EXISTS AN ELEMENT a IN G SUCH THAT $\psi(x) = a\,\phi(x)$ IS A HOMOMORPHISM (THAT IS, $\psi(xy) = \psi(x)\psi(y)$ FOR ALL x AND y IN G).

IF THERE IS SUCH AN a, THEN IT MUST BE $\phi(e)^{-1}$ BECAUSE

$$e = \psi(e) = a\,\phi(e) \Rightarrow a = \phi(e)^{-1}.$$

WE SHOW THAT $\psi(x) = \phi(e)^{-1}\phi(x)$ IS A HOMOMORPHISM, I.E.,

$$\psi(xy) = \psi(x)\psi(y),$$

I.E.,

$$\phi(e)^{-1}\phi(xy) = \phi(e)^{-1}\phi(x)\phi(e)^{-1}\phi(y)$$

I.E.,

$$\phi(xy) = \phi(x)\phi(e)^{-1}\phi(y)$$

TO PROVE THIS NOTE THAT

$$\phi(x)\phi(y)\phi(y^{-1}x^{-1}) = \phi(e)\phi(xy)\phi(y^{-1}x^{-1})$$

SO

$$\phi(x)\phi(y) = \phi(e)\phi(xy)$$

AND THEREFORE

$$\phi(xy) = \phi(e)^{-1}\phi(x)\phi(y)$$

BUT $\phi(x)\phi(e)\phi(x^{-1}) = \phi(e)\phi(x)\phi(x^{-1}) \Rightarrow \phi(x)\phi(e) = \phi(e)\phi(x)$

$\Rightarrow \phi(e)^{-1}\phi(x) = \phi(x)\phi(e)^{-1}$

$$\phi(xy) = \phi(x)\phi(e)^{-1}\phi(y)$$

AS REQUIRED.