PROBLEM SOLVING SEMINAR

PROBLEM SET 7 (SOLUTIONS)

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1. EVALUATE THE SUM

$$\sum_{j=0}^{n} \sum_{i=j}^{n} \binom{n}{i}\binom{i}{j}$$

$$\sum_{j=0}^{n} \sum_{i=j}^{n} \binom{n}{i}\binom{i}{j} = \sum_{i=0}^{n} \binom{n}{i}\binom{i}{0} + \sum_{i=1}^{n} \binom{n}{i}\binom{i}{1} + \cdots + \sum_{i=n}^{n} \binom{n}{i}\binom{i}{n}$$

$$= \binom{n}{0}\binom{0}{0} + \binom{n}{1}\binom{1}{0} + \cdots + \binom{n}{n}\binom{n}{0} +$$

$$\binom{n}{1}\binom{1}{1} + \cdots + \binom{n}{n}\binom{n}{1} + \cdots +$$

$$\binom{n}{n}\binom{n}{n}$$

$$= \binom{n}{0}\binom{0}{0} + \binom{n}{1}\left[\binom{1}{0}+\binom{1}{1}\right] + \binom{n}{2}\left[\binom{2}{0}+\binom{2}{1}+\binom{2}{2}\right] + \cdots +$$

$$\binom{n}{n}\left[\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}\right]$$

$$= \binom{n}{0}\sum_{j=0}^{0}\binom{0}{j} + \binom{n}{1}\sum_{j=0}^{1}\binom{1}{j} + \binom{n}{2}\sum_{j=0}^{2}\binom{2}{j} + \cdots + \binom{n}{n}\sum_{j=0}^{n}\binom{n}{j}$$

TO OBTAIN THE $\sum_{j=0}^{i}\binom{i}{j}$ WE USE THE BINOMIAL THEOREM

$$(1+x)^i = \sum_{j=0}^{i}\binom{i}{j}x^j$$

WITH $x=1$:

$$2^i = \sum_{j=0}^{i}\binom{i}{j}.$$

THUS,

$$\sum_{j=0}^{n} \sum_{i=j}^{n} \binom{n}{i}\binom{i}{j} = \binom{n}{0}2^0 + \binom{n}{1}2^1 + \binom{n}{2}2^2 + \cdots + \binom{n}{n}2^n$$

$$= \sum_{i=0}^{n} \binom{n}{i} 2^i$$

$$= (1+2)^n$$

$$= 3^n .$$

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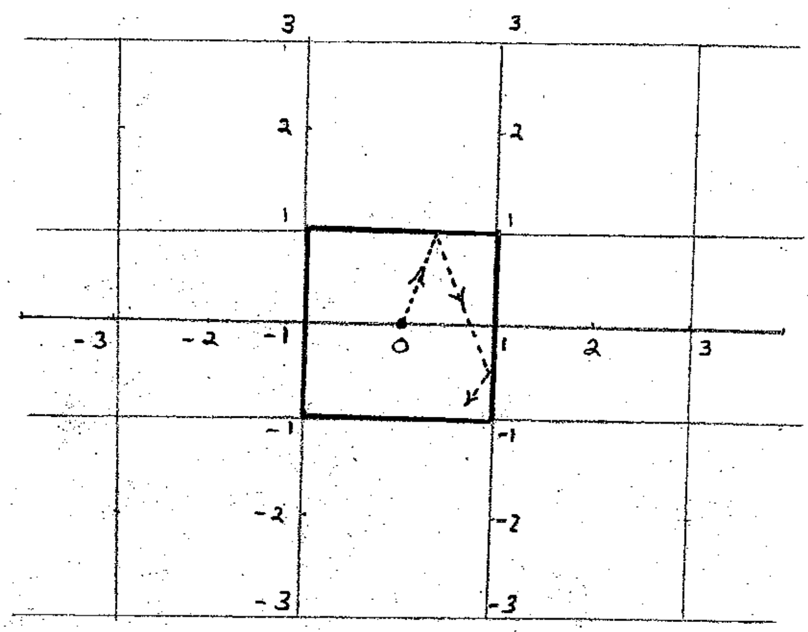
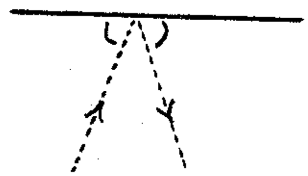
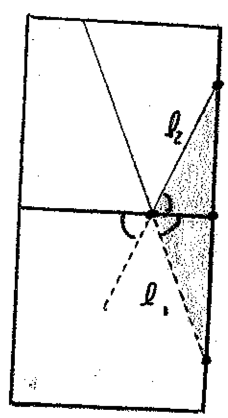
2. ON A SMALL SQUARE BILLIARD TABLE WITH SIDES OF LENGTH 2 ft A BALL IS PLAYED FROM THE CENTER AND, AFTER REBOUNDING OFF THE SIDES SEVERAL TIMES, GOES INTO A CUP AT ONE OF THE CORNERS. PROVE THAT THE TOTAL DISTANCE TRAVELLED BY THE BALL IS <u>NOT</u> AN INTEGER NUMBER OF FEET.



NOTE: YOU CAN'T REALLY EXPECT TO SOLVE A PROBLEM LIKE THIS UNLESS YOU KNOW (OR ASSUME) SOMETHING ABOUT HOW BILLIARD BALLS BOUNCE OF THE EDGE OF A BILLIARD TABLE. THE RELEVANT FACT IS THAT THE ANGLE AT WHICH IT STRIKES THE WALL IS THE SAME AS THE ANGLE AT WHICH IT IS REFLECTED FROM THE WALL.



CONSEQUENTLY, THE LENGTHS ℓ_1 AND ℓ_2 BELOW ARE EQUAL (CONGRUENT RIGHT TRIANGLES).



THUS, WE CAN REPHRASE OUR PROBLEM AS FOLLOWS: CONSIDER A GRID IN THE XY-PLANE WITH HORIZONTAL LINES AT $y = 2n+1$ AND VERTICAL LINES AT $x = 2n+1$, WHERE n IS AN ARBITRARY INTEGER. A BALL STARTS AT THE ORIGIN AND TRAVELS IN A STRAIGHT LINE UNTIL IT REACHES A POINT OF INTERSECTION OF A HORIZONTAL LINE AND A VERTICAL LINE OF THE GRID. THE CLAIM IS THAT THE DISTANCE d TRAVELLED BY THE BALL IS NOT AN INTEGER. HOWEVER, FOR SOME

INTEGERS n AND m WE HAVE

$$d^2 = (2n+1)^2 + (2m+1)^2$$
$$= 4(n^2+m^2) + 4(n+m) + 2$$

SO, IF d IS AN INTEGER, THIS WOULD IMPLY THAT $d^2 \equiv 2 \bmod 4$. BUT ANY SQUARE IS $\equiv \bmod 4$ TO 0 OR 1 SO THIS IS A CONTRADICTION AND d CANNOT BE AN INTEGER.

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3. COMPUTE $a_0 + a_1 + \cdots + a_{203}$ IF

$$a_0 = 2$$
$$a_1 = 5$$
$$a_n = 5a_{n-1} - 6a_{n-2}, \quad n \geq 2$$

WE FIRST FIND A CLOSED FORM EXPRESSION FOR a_n VIA THE GENERATING FUNCTION $A(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots$

FOR $n \geq 2$,

$$a_n = 5a_{n-1} - 6a_{n-2} \Rightarrow a_n x^n = 5a_{n-1}x^n - 6a_{n-2}x^n$$
$$= 5x\, a_{n-1}x^{n-1} - 6x^2 a_{n-2} x^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n = 5x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} - 6x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2}$$
$$= 5x \sum_{n=1}^{\infty} a_n x^n - 6x^2 \sum_{n=0}^{\infty} a_n x^n$$

$\Rightarrow$

$$A(x) - a_0 - a_1 x = 5x\,(A(x) - a_0) - 6x^2 A(x)$$

$$A(x) - 2 - 5x = 5x\,(A(x) - 2) - 6x^2 A(x)$$

SOLVING FOR $A(x)$ GIVES

$$A = \frac{2-5x}{1-5x-6x^2} = \frac{2-5x}{(1-2x)(1-3x)} = \frac{1}{1-2x} + \frac{1}{1-3x}$$

(PARTIAL FRACTIONS)

$$= \sum_{n=0}^{\infty} (2x)^n + \sum_{n=0}^{\infty} (3x)^n$$

$$= \sum_{n=0}^{\infty} (2^n + 3^n)\,x^n$$

THUS,

$$a_n = 2^n + 3^n \,, \quad n \geq 0.$$

NOW, FOR ANY $k \geq 0$,

$$a_0 + a_1 + \cdots + a_k = (2^0+3^0) + (2^1+3^1) + \cdots + (2^k+3^k)$$

$$= (2^0+2^1+\cdots+2^k) + (3^0+3^1+\cdots+3^k)$$

$$= \frac{2^{k+1}-1}{2-1} + \frac{3^{k+1}-1}{3-1}$$

$$= \frac{2^{k+2}+3^{k+1}-3}{2}$$

IN PARTICULAR,

$$a_0 + a_1 + \cdots + a_{203} = \frac{2^{205}+3^{204}-3}{2}$$

THERE IS ANOTHER WAY TO APPROACH SUCH RECURRENCE RELATIONS THAT WE WILL NOW BRIEFLY DESCRIBE WITHOUT PROVIDING ANY DETAILS.

NOW WE WILL ARRIVE AT THE SAME RESULT USING A GENERAL FACT ABOUT RECURRENCES.

IF $c_1, \dots, c_d$ ARE CONSTANTS, THEN

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d}$$

IS A "LINEAR, HOMOGENEOUS RELATION OF ORDER d". ITS "CHARACTERISTIC POLYNOMIAL" IS

$$p(t) = t^d - c_1 t^{d-1} - c_2 t^{d-2} - \cdots - c_d .$$

SUPPOSE THE ROOTS $\lambda_1, \dots, \lambda_d$ OF $p(t) = 0$ ARE ALL DISTINCT. THEN EACH OF

$$a_n = \lambda_1^n$$
$$\vdots$$
$$a_n = \lambda_d^n,$$

IS A SOLUTION TO THE RECURRENCE RELATION AND, MOREOVER, <u>EVERY</u> SOLUTION IS A LINEAR COMBINATION

$$\alpha_1 \lambda_1^n + \cdots + \alpha_d \lambda_d^n$$

OF THESE.

FOR THE RECURRENCE RELATION IN THIS PROBLEM ($a_n = 5a_{n-1} - 6a_{n-2}$) THE CHARACTERISTIC POLYNOMIAL IS

$$p(t) = t^2 - 5t + 6 = (t-2)(t-3)$$

SO THE ROOTS ARE $\lambda_1 = 2$ AND $\lambda_2 = 3$. THUS, FOR SOME α_1, α_2,

$$a_n = \alpha_1 2^n + \alpha_2 3^n$$

SINCE $a_0 = 2$ AND $a_1 = 5$,

$$2 = \alpha_1 + \alpha_2$$
$$5 = \alpha_1 \cdot 2 + \alpha_2 \cdot 3$$

FROM WHICH WE OBTAIN $\alpha_1 = 1$ AND $\alpha_2 = 1$ SO

$$\alpha_n = 2^n + 3^n$$

AS BEFORE.

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4. **PROVE THAT THERE ARE NO PRIMES IN THE SEQUENCE**

10001, 100010001, 1000100010001, ...

THE TERMS OF THE SEQUENCE ARE

$$1 + 10^4$$
$$1 + 10^4 + 10^8$$
$$1 + 10^4 + 10^8 + 10^{12}$$
$$\vdots$$
$$1 + 10^4 + 10^8 + 10^{12} + \cdots + 10^{4n}$$
$$\vdots$$

SUPPOSE FIRST THAT $n = 2k$, $k \geq 1$, IS EVEN. THEN

$$1+10^4+10^8+\cdots+10^{4(2k)} = 1+(10^4)^1+(10^4)^2+\cdots+(10^4)^{2k}$$
$$= \frac{(10^4)^{2k+1}-1}{10^4-1}$$
$$= \frac{(10^2)^{2k+1}-1}{10^2-1}\ \frac{(10^2)^{2k+1}+1}{10^2+1}$$
$$= (1+10^2+\cdots+(10^2)^{2k})(1-10^2+(10^2)^2-\cdots+(10^2)^{2k})$$

WHICH IS COMPOSITE. NOW SUPPOSE $n=2k+1$, $k \geq 0$, IS ODD. THEN

$$1+10^4+10^8+10^{12}+\cdots+10^{4(2k+1)} = 1+10^4+10^8+10^8 10^4+\cdots+10^{8k}+10^{8k}10^4$$
$$= (1+10^4)+10^8(1+10^4)+\cdots+10^{8k}(1+10^4)$$
$$= (1+10^4)(1+10^8+\cdots+10^{8k})$$

WHICH IS CLEARLY COMPOSITE FOR $k>0$. CHECKING $k=0$ SEPARATELY GIVES $1+10^4 = 10001 = 73 \cdot 137$ SO THIS TOO IS COMPOSITE.

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5. PROVE THAT THERE EXISTS A UNIQUE FUNCTION f FROM THE SET $\mathbb{R}^+$ OF POSITIVE REAL NUMBERS TO $\mathbb{R}^+$ SUCH THAT

$$f(f(x)) = 6x - f(x)$$

NOTICE THAT $f(x)=2x$ SATISFIES THESE CONDITIONS. WE SHOW THAT IT IS THE ONLY SUCH FUNCTION.

BECAUSE $f(f(x)) = 6x - f(x)$ IS AN ITERATIVE CONDITION WE WILL REFORMULATE THE PROBLEM IN TERMS OF A RECURRENCE RELATION.

FIX SOME ARBITRARY $X > 0$. DEFINE A SEQUENCE OF POSITIVE NUMBERS $(a_n)_{n=0}^{\infty}$ BY $a_0 = X$ AND $a_{n+1} = f(a_n)$, $n \geq 0$. THUS, $f(f(x)) = 6x - f(x)$ BECOMES $a_2 = 6a_0 - a_1$ AND, BY INDUCTION,

$$a_{n+2} + a_{n+1} - 6a_n = 0, \quad n \geq 0.$$

THUS, WE NEED ONLY DETERMINE a_1 IN TERMS OF a_0. WE SOLVE THE RECURRENCE FIRST WITH A GENERATING FUNCTION AND THEN STATE A GENERAL RESULT (PROVABLE WITH GENERATING FUNCTIONS) FROM WHICH THE SOLUTION CAN BE OBTAINED MORE QUICKLY.

LET $A(x) = \sum_{n=0}^{\infty} a_n x^n$ BE THE GENERATING FUNCTION.

$$a_{n+2} + a_{n+1} - 6a_n = 0$$

$$a_{n+2}x^n + a_{n+1}x^n - 6a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}x^n + \sum_{n=0}^{\infty} a_{n+1}x^n - 6\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\frac{A(x) - (a_0 + a_1 x)}{x^2} + \frac{A(x) - a_0}{x} - 6A(x) = 0$$

$$A(x) - a_0 - a_1 x + xA(x) - a_0 x - 6x^2 A(x) = 0$$

$$(-6x^2 + x + 1)A(x) = a_0 + (a_0 + a_1)x$$

$$A(x) = -\frac{(a_0+a_1)x + a_0}{6x^2 - x - 1} = -\frac{(a_0+a_1)x + a_0}{(2x-1)(3x+1)}$$

$$= -\left[\frac{\frac{3}{5}a_0 + \frac{1}{5}a_1}{2x-1} + \frac{-\frac{2}{5}a_0 + \frac{1}{5}a_1}{3x+1}\right]$$

(PARTIAL FRACTIONS

$$= \left(\tfrac{3}{5}a_0 + \tfrac{1}{5}a_1\right)\frac{1}{1-2x} +$$

$$\left(\tfrac{2}{5}a_0 - \tfrac{1}{5}a_1\right)\frac{1}{1-(-3x)}$$

$$= \left(\tfrac{3}{5}a_0 + \tfrac{1}{5}a_1\right)\sum_{n=0}^{\infty} 2^n x^n$$

$$\left(\tfrac{2}{5}a_0 - \tfrac{1}{5}a_1\right)\sum_{n=0}^{\infty} (-3)^n x^n$$

$$= \sum_{n=0}^{\infty}\left[\left(\tfrac{3}{5}a_0 + \tfrac{1}{5}a_1\right)2^n + \left(\tfrac{2}{5}a_0 - \tfrac{1}{5}a_1\right)(-3)^n\right]x^n$$

THUS, FOR $n \geq 0$, a_n HAS THE FORM $\alpha_1(2^n) + \alpha_2(-3)^n$ AND, IF α_2 WERE NOT ZERO, THIS HAS THE SAME SIGN AS $\alpha_2(-3)^n$ FOR LARGE n, WHICH ALTERNATES AND THIS CONTRADICTS $a_n = f(a_{n-1}) > 0$. THUS, $\alpha_2 = 0$, I.E., $\frac{2}{5}a_0 - \frac{1}{5}a_1 = 0$ SO

$$a_1 = 2a_0$$

$$f(x) = 2x.$$

<u>NOTE</u>: ONE COULD USE INSTEAD THE RESULTS ON LINEAR RECURRENCE RELATIONS DESCRIBED IN THE SOLUTION TO PROBLEM #3.

FOR OUR RECURRENCE $a_n = -a_{n-1} + 6a_{n-2}$, $p(t) = t^2 + t - 6$ AND THE ZEROS ARE 2 AND -3 SO EVERY SOLUTION IS OF THE FORM

$$\alpha_1(2^n) + \alpha_2(-3)^n.$$

AS ABOVE, α_2 MUST BE 0 SO $a_n = \alpha_1(2^n)$. THUS, $a_0 = \alpha_1$ AND $a_1 = \alpha_1(2^1) = 2\alpha_1 = 2a_0$ SO $f(x) = 2x$.