PROBLEM SOLVING SEMINAR

PROBLEM SET 8 (SOLUTIONS)

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1. THE VTRC BUS COMPANY SERVES CITIES IN THE USA. A SUBSET S OF THESE CITIES IS CALLED WELL-SERVED IF IT HAS AT LEAST 3 CITIES AND IF, FROM ANY CITY A IN S, ONE CAN TAKE A NONSTOP VTRC BUS TO TWO DIFFERENT OTHER CITIES B AND C IN S (ALTHOUGH THERE IS NOT NECESSARILY A NONSTOP BUS FROM B TO A OR FROM C TO A). SUPPOSE THERE IS A WELL-SERVED SUBSET S. PROVE THAT THERE IS A WELL-SERVED SUBSET T SUCH THAT, FOR ANY TWO CITIES A, B IN T, ONE CAN TRAVEL BY VTRC BUS FROM A TO B, STOPPING ONLY IN CITIES IN T.

BY ASSUMPTION, THERE IS A WELL-SERVED SET S. FOR EACH $x \in S$ LET S_x BE THE SET OF ALL CITIES TO WHICH YOU CAN TRAVEL ON VTRC BUS FROM x (INCLUDING x). SINCE S IS WELL-SERVED, SO IS EACH G_x. THERE ARE ONLY FINITELY MANY SUCH G_x SO WE CAN SELECT A G_x FOR WHICH $|G_x|$ (THE NUMBER OF ELEMENTS IN G_x) IS MINIMAL. NOW LET $y, z \in G_x$. WE MUST SHOW THAT ONE CAN TRAVEL FROM y TO x STOPPING ONLY AT CITIES ONLY IN G_x. IT IS CLEARLY ENOUGH TO PROVE THIS WHEN $z = x$. SUPPOSE IT IS NOT POSSIBLE TO TRAVEL FROM y TO x STOPPING ONLY

AT CITIES IN G_x. SINCE $y \in G_x$, $G_y \subseteq G_x$. BUT, BY ASSUMPTION, $x \notin G_y$ SO $|G_y| < |G_x|$ AND THIS CONTRADICTS THE MINIMALITY OF $|G_x|$.

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2. SHOW THAT FOR ANY SET OF n INTEGERS $(n \geq 1)$ THERE IS A SUBSET OF THEM WHOSE SUM IS DIVISIBLE BY n.

LET $x_1, \ldots, x_n$ BE THE GIVEN INTEGERS. CONSIDER

$$y_1 = x_1$$
$$y_2 = x_1 + x_2$$
$$\vdots$$
$$y_n = x_1 + x_2 + \cdots + x_n$$

IF SOME $y_i \equiv 0 \bmod n$ WE ARE DONE SO SUPPOSE THIS IS NOT THE CASE. THEN $y_1, \ldots, y_n$ ARE ALL CONGRUENT MOD n TO ONE OF $1, \ldots, n-1$. BY THE PIGEONHOLE PRINCIPLE AT LEAST TWO OF $y_1, \ldots, y_n$ MUST BE CONGRUENT MOD n. SUPPOSE

$$y_i \equiv y_j \bmod n$$

WITH $i < j$. THEN $y_j - y_i \equiv 0 \bmod n$. BUT

$$y_j - y_i = x_{i+1} + \cdots + x_j$$

SO $x_{i+1} + \cdots + x_j$ IS DIVISIBLE BY n.

3. LET $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ BE NON-NEGATIVE REAL NUMBERS. PROVE THAT

$$\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n} b_i\right)^{\frac{1}{n}} \leq \left(\prod_{i=1}^{n} (a_i + b_i)\right)^{\frac{1}{n}}$$

WE USE THE ARITHMETIC-GEOMETRIC MEAN INEQUALITY : $X_1, \ldots, X_n$ NON-NEGATIVE $\Rightarrow \sqrt[n]{X_1 \cdots X_n} \leq \dfrac{X_1 + \cdots + X_n}{n}$. IF ANY a_i OR b_i IS ZERO THE RESULT IS OBVIOUS SO WE CAN ASSUME THEY ARE ALL POSITIVE AND REWRITE THE INEQUALITY AS $\left(\prod_{i=1}^{n} \frac{a_i}{a_i+b_i}\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n} \frac{b_i}{a_i+b_i}\right)^{\frac{1}{n}} \leq 1$, BUT

$$\left(\prod_{i=1}^{n} \frac{a_i}{a_i+b_i}\right)^{\frac{1}{n}} \leq \frac{\sum_{i=1}^{n} \frac{a_i}{a_i+b_i}}{n}$$

AND

$$\left(\prod_{i=1}^{n} \frac{b_i}{a_i+b_i}\right)^{\frac{1}{n}} \leq \frac{\sum_{i=1}^{n} \frac{b_i}{a_i+b_i}}{n}$$

ADDING THE TWO INEQUALITIES GIVES

$$\left(\prod_{i=1}^{n} \frac{a_i}{a_i+b_i}\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n} \frac{b_i}{a_i+b_i}\right)^{\frac{1}{n}} \leq \frac{1}{n}\left(\sum_{i=1}^{n} \frac{a_i}{a_i+b_i} + \sum_{i=1}^{n} \frac{b_i}{a_i+b_i}\right) =$$

$$\frac{1}{n}\left(\sum_{i=1}^{n} \frac{a_i+b_i}{a_i+b_i}\right) = \frac{1}{n}(n) = 1$$

4. LET A AND B BE 2×2 MATRICES WITH INTEGER ENTRIES SUCH THAT A, $A+B$, $A+2B$, $A+3B$ AND $A+4B$ ARE ALL INVERTIBLE MATRICES WHOSE INVERSES HAVE INTEGER ENTRIES. SHOW THAT $A+5B$ IS INVERTIBLE AND THAT ITS INVERSE HAS INTEGER ENTRIES.

FIRST NOTE THAT A SQUARE MATRIX WITH INTEGER ENTRIES HAS AN INVERSE WITH INTEGER ENTRIES IF AND ONLY IF ITS DETERMINANT IS ± 1 (IF M HAS INTEGER ENTRIES AND ITS INVERSE DOES AS WELL, THEN $(\det M)(\det M^{-1}) = \det(MM^{-1}) = 1$ AND BOTH DETERMINANTS ARE INTEGERS SO EITHER BOTH ARE 1 OR BOTH ARE -1; CONVERSELY, IF M HAS INTEGER ENTRIES AND $\det M = \pm 1$, THEN ITS INVERSE IS $\pm\, \mathrm{adj}(M)$ AND THIS HAS INTEGER ENTRIES.)

NOW LET A AND B BE AS IN THE PROBLEM. FOR ANY $n = 0,1,2,\ldots$ CONSIDER $\det(A+nB)$. SINCE A AND B ARE 2×2 MATRICES THIS IS A POLYNOMIAL OF DEGREE AT MOST 2 IN n, I.E.,

$\det(A+nB) = \alpha_0 + \alpha_1 n + \alpha_2 n^2$ FOR SOME INTEGERS α_0, α_1 AND α_2. BUT THIS POLYNOMIAL TAKES THE VALUES ± 1 FOR $n = 0,1,2,3,4$ SO THE PIGEONHOLE PRINCIPLE IMPLIES THAT IT MUST TAKE ONE OF THESE VALUES AT LEAST 3 TIMES. BEING AT MOST QUADRATIC, THIS IS POSSIBLE ONLY IF THE POLYNOMIAL IS CONSTANT. IN PARTICULAR, $\det(A+nB) = \pm 1$ FOR ALL $n = 0,1,2,\ldots$ SO EVERY $A+nB$ HAS AN INVERSE WITH INTEGER ENTRIES.

5. LET a, b AND c BE POSITIVE REAL NUMBERS AND SUPPOSE THAT

$$a\cos^2\theta + b\sin^2\theta < c.$$

SHOW THAT

$$\sqrt{a}\cos^2\theta + \sqrt{b}\sin^2\theta < \sqrt{c}.$$

CONSIDER

$$(\sqrt{a}\cos^2\theta + \sqrt{b}\sin^2\theta)^2 =$$

$$a^2\cos^4\theta + \underbrace{2\sqrt{a}\sqrt{b}}_{2\sqrt{ab}\ \le\ a+b}\cos^2\theta\sin^2\theta + b\sin^4\theta$$

(ARITHMETIC-GEOMETRIC MEAN INEQUALITY)

$$\le a^2\cos^4\theta + (a+b)\cos^2\theta\sin^2\theta + b\sin^4\theta =$$

$$(a\cos^2\theta + b\sin^2\theta)(\cos^2\theta + \sin^2\theta) =$$

$$a\cos^2\theta + b\sin^2\theta < c$$

THUS,

$$(\sqrt{a}\cos^2\theta + \sqrt{b}\sin^2\theta)^2 < c$$

SO TAKING SQUARE ROOTS GIVES THE RESULT.