

PROBLEM SOLVING SEMINAR

PROBLEM SET 8

1. THE VTRC BUS COMPANY SERVES CITIES IN THE USA. A SUBSET  $S$  OF THESE CITIES IS CALLED WELL-SERVED IF IT HAS AT LEAST 3 CITIES AND IF, FROM ANY CITY  $A$  IN  $S$ , ONE CAN TAKE A NONSTOP VTRC BUS TO TWO DIFFERENT OTHER CITIES  $B$  AND  $C$  IN  $S$  (ALTHOUGH THERE IS NOT NECESSARILY A NONSTOP BUS FROM  $B$  TO  $A$  OR FROM  $C$  TO  $A$ ). SUPPOSE THERE IS A WELL-SERVED SUBSET  $S$ . PROVE THAT THERE IS A WELL-SERVED SUBSET  $T$  SUCH THAT, FOR ANY TWO CITIES  $A, B$  IN  $T$ , ONE CAN TRAVEL BY VTRC BUS FROM  $A$  TO  $B$ , STOPPING ONLY IN CITIES IN  $T$ .

2. SHOW THAT, FOR ANY SET OF  $n$  INTEGERS ( $n \geq 1$ ) THERE IS A SUBSET OF THEM WHOSE SUM IS DIVISIBLE BY  $n$ .

3. LET  $a_1, \dots, a_n, b_1, \dots, b_n$  BE NON-NEGATIVE REAL NUMBERS. PROVE THAT

$$\left( \prod_{i=1}^n a_i \right)^{\frac{1}{n}} + \left( \prod_{i=1}^n b_i \right)^{\frac{1}{n}} \leq \left( \prod_{i=1}^n (a_i + b_i) \right)^{\frac{1}{n}}$$

4. LET  $A$  AND  $B$  BE  $2 \times 2$  MATRICES WITH INTEGER ENTRIES SUCH THAT  $A, A+B, A+2B, A+3B$ , AND  $A+4B$  ARE ALL INVERTIBLE MATRICES WHOSE INVERSES HAVE INTEGER ENTRIES. SHOW THAT  $A+5B$  IS INVERTIBLE AND THAT ITS INVERSE HAS INTEGER ENTRIES.

5. LET  $a, b$  AND  $c$  BE POSITIVE REAL NUMBERS AND SUPPOSE THAT

$$a \cos^2 \theta + b \sin^2 \theta < c.$$

SHOW THAT

$$\sqrt{a} \cos^2 \theta + \sqrt{b} \sin^2 \theta < \sqrt{c}.$$