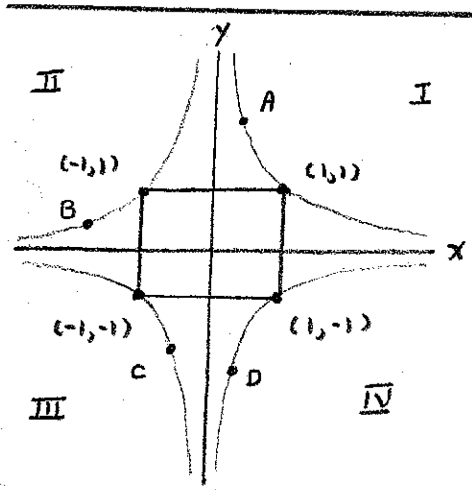
PROBLEM SET 9 (SOLUTIONS)

1. FIND THE LEAST POSSIBLE AREA OF A CONVEX SET IN THE PLANE THAT INTERSECTS BOTH BRANCHES OF THE HYPERBOLA $xy=1$ AND BOTH BRANCHES OF THE HYPERBOLA $xy=-1$. (A SET S IN THE PLANE IS CALLED CONVEX IF FOR ANY TWO POINTS IN S THE LINE SEGMENT CONNECTING THEM IS CONTAINED IN S.)



THE AREA OF THE SQUARE WITH VERTICES $(\pm 1, \pm 1)$ IS 4 AND WE SHOW THAT THIS IS THE SMALLEST POSSIBLE.

LET S BE A CONVEX SET OF THE SORT DESCRIBED, CHOOSE POINTS A, B, C, D IN S IN QUADRANTS I, II, III, IV AND LYING ON A BRANCH OF ONE OF THE HYPERBOLAS. LET $A=(a,\frac{1}{a})$, $B=(-b,\frac{1}{b})$, $C=(-c,-\frac{1}{c})$ AND $D=(d,-\frac{1}{d})$, WHERE $a,b,c,d>0$. SINCE S IS CONVEX IT CONTAINS THE QUADRILATERAL $ABCD$ SO THE AREA OF S GREATER THAN OR EQUAL TO THE AREA OF $ABCD$ WHICH WE FIND FROM $\frac{1}{2}\|\vec{AB}\times\vec{AD}\| + \frac{1}{2}\|\vec{CD}\times\vec{CB}\|$:

$$\vec{AB} = \langle -b-a, \tfrac{1}{b}-\tfrac{1}{a}, 0\rangle \qquad \vec{AD} = \langle d-a, -\tfrac{1}{d}-\tfrac{1}{a}, 0\rangle$$

$$\vec{AB}\times\vec{AD} = \langle 0, 0, \tfrac{b}{d}-\tfrac{d}{b}+\tfrac{a}{b}+\tfrac{b}{a}+\tfrac{a}{d}+\tfrac{d}{a}\rangle$$

AND, SIMILARLY,

$$\vec{CD}\times\vec{CB} = \langle 0, 0, -\tfrac{b}{d}+\tfrac{d}{b}+\tfrac{c}{d}+\tfrac{d}{c}+\tfrac{b}{c}+\tfrac{c}{b}\rangle$$

BOTH $\vec{AB} \times \vec{AD}$ AND $\vec{CD} \times \vec{CB}$ HAVE POSITIVE Z-COMPONENT (BY THE RIGHT-HAND RULE) SO

$$\text{AREA OF } ABCD = \frac{1}{2}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} + \frac{b}{a} + \frac{c}{b} + \frac{d}{c} + \frac{a}{d}\right)$$

$$= \frac{1}{2}\left(\frac{a}{b} + \frac{b}{c}\right) + \frac{1}{2}\left(\frac{c}{d} + \frac{d}{a}\right) + \frac{1}{2}\left(\frac{b}{a} + \frac{c}{b}\right) + \frac{1}{2}\left(\frac{d}{c} + \frac{a}{d}\right)$$

$$\geq \sqrt{\frac{a}{b}\frac{b}{c}} + \sqrt{\frac{c}{d}\frac{d}{a}} + \sqrt{\frac{b}{a}\frac{c}{b}} + \sqrt{\frac{d}{c}\frac{a}{d}}$$

(ARITHMETIC - GEOMETRIC MEAN INEQUALITY)

$$\text{AREA OF } ABCD \geq 2\left(\sqrt{\frac{a}{c}} + \sqrt{\frac{c}{a}}\right)$$

$$\geq 2(2) = 4$$

BECAUSE THE FUNCTION

$$f(x) = \sqrt{x} + \sqrt{\frac{1}{x}} = x^{\frac{1}{2}} + x^{-\frac{1}{2}},$$

$$x > 0$$

ACHIEVES A MINIMUM VALUE OF 2 AT $x = 1$.

THUS,

$$\text{AREA OF } S \geq 4$$

2. LET k BE A POSITIVE INTEGER. THE n^{TH} DERIVATIVE OF $\frac{1}{x^k-1}$ HAS THE FORM $\frac{P_n(x)}{(x^k-1)^{n+1}}$, WHERE $P_n(x)$ IS A POLYNOMIAL. FIND $P_n(1)$.

THE $(n+1)^{ST}$ DERIVATIVE OF $\frac{1}{x^k-1}$ IS

$$\frac{d}{dx}\left(\frac{P_n(x)}{(x^k-1)^{n+1}}\right) = \frac{(x^k-1)P_n'(x) - (n+1)kx^{k-1}P_n(x)}{(x^k-1)^{n+2}}$$

SO

$$P_{n+1}(x) = (x^k-1)P_n'(x) - (n+1)kx^{k-1}P_n(x)$$

AND

$$P_{n+1}(1) = -(n+1)k\,P_n(1)$$

$n=1$: $\left(\frac{1}{x^k-1}\right)' = \frac{-kx^{k-1}}{(x^k-1)^2} \Rightarrow P_1(1) = -k$

THUS,

$$P_2(1) = -2kP_1(1) = (-1)^2 2k^2$$
$$P_3(1) = -3kP_2(1) = (-1)^3 3!\,k^3$$
$$\vdots$$

<u>CLAIM</u>: $P_n(1) = (-1)^n n!\,k^n$ FOR $n \geq 1$

<u>PROOF</u>: $P_1(1) = -k = (-1)^1 1!\,k^1$. NOW ASSUME $m \geq 1$ AND

$$P_m(1) = (-1)^m m!\,k^m$$

THEN

$$P_{m+1}(1) = -(m+1)kP_m(1) = (-1)^{m+1}(m+1)!\,k^{m+1} \qquad \square$$

3. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a surjective map with the property that if the points A, B and C are collinear, then so are $f(A)$, $f(B)$ and $f(C)$. Prove that f is bijective.

Suppose f is not bijective. Since f is surjective, this means that there is a point $A_0 \in \mathbb{R}^2$ such that at least two distinct points are mapped to A_0 by f. Choose points $B_0, C_0 \in \mathbb{R}^2$ such that A_0, B_0, C_0 are not collinear. Now select points $A, B, C \in \mathbb{R}^2$ such that $f(A) = A_0$, $f(B) = B_0$ and $f(C) = C_0$. Since f maps collinear points to collinear points, we see that A, B, C are not collinear. Now given two sets each with three non-collinear points, there is a bijective affine transformation (i.e. a linear map composed with a translation) sending the first set of points to the second set. This means that there are bijective affine transformations $g, h\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $g(0,0) = A$, $g(0,1) = B$, $g(1,0) = C$, $h(A_0) = (0,0)$, $h(B_0) = (0,1)$, $h(C_0) = (1,0)$. Then $k := hfg\colon \mathbb{R}^2 \to \mathbb{R}^2$ fixes $(0,0), (0,1), (1,0)$, and has the property that if P, Q, R are collinear, the so are $k(P), k(Q), k(R)$. Also there is a point $(a,b) \neq (0,0)$ such that $k(a,b) = (0,0)$. We want to show that this situation cannot happen.

Without loss of generality, we may assume that $b \neq 0$ Let ℓ denote the line joining $(1,0)$ to $(0,b)$. Then $k(\ell)$ is contained in the x-axis. We claim that k maps the horizontal line through $(0,1)$ into itself. For if this was not the case, there would be a point with coordinates (c,d) with $d \neq 1$ such that $k(c,d) = (1,1)$. Then if m was the line joining (c,d) to $(0,1)$, we would have $k(m)$ contained in the horizontal line through $(0,1)$. Since m intersects the x-axis and the x-axis is mapped into itself by k, this is not possible and so our claim is established. Now let ℓ meet this horizontal line at the point P. Then we have that $k(P)$ is both on this horizontal line and also the x-axis, a contradiction and the result follows.

4. LET N_n DENOTE THE NUMBER OF ORDERED n-TUPLES OF POSITIVE INTEGERS $(a_1, a_2, \ldots, a_n)$ SUCH THAT

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = 1,$$

DETERMINE WHETHER N_{10} IS EVEN OR ODD.

WE DETERMINE THE PARITY OF N_{10} AS FOLLOWS. SINCE THE 10-TUPLES ARE ASSUMED <u>ORDERED</u>, IF $(a_1, a_2, \ldots, a_{10})$ IS A SOLUTION WITH $a_1 \neq a_2$, THEN $(a_2, a_1, \ldots, a_{10})$ IS ANOTHER SOLUTION. THUS, SOLUTIONS WITH $a_1 \neq a_2$ OCCUR IN PAIRS SO DELETING THEM FROM THE SET OF SOLUTIONS DOES NOT CHANGE ITS PARITY, I.E., WE CAN CONSIDER ONLY SOLUTIONS WITH $a_1 = a_2$. IN THE SAME WAY WE CAN RESTRICT ATTENTION TO SOLUTIONS WITH $a_3 = a_4$, $a_5 = a_6$, $a_7 = a_8$ AND $a_9 = a_{10}$. THESE SATISFY

$$\frac{2}{a_1} + \frac{2}{a_3} + \frac{2}{a_5} + \frac{2}{a_7} + \frac{2}{a_9} = 1$$

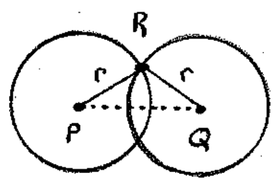
BUT NOW WE CAN RESTRICT ATTENTION TO SOLUTIONS WITH $a_1 = a_3$, AND $a_5 = a_7$ SO THAT

$$\frac{4}{a_1} + \frac{4}{a_5} + \frac{2}{a_9} = 1.$$

ONCE MORE, WE CAN ASSUME $a_1 = a_5$ SO $\frac{8}{a_1} + \frac{2}{a_9} = 1$, I.E., $a_1 a_9 - 8a_9 - 2a_1 = 0 \Rightarrow a_9(a_1 - 8) - 2(a_1 - 8) = 16 \Rightarrow$ $(a_1 - 8)(a_9 - 2) = 16$ WHICH HAS 5 SOLUTIONS OBTAINED BY WRITING 16 AS $16 \cdot 1$, $8 \cdot 2$, $4 \cdot 4$, $2 \cdot 8$, $1 \cdot 16$. THUS, N_{10} IS ODD.

5. CONSIDER A PAPER PUNCH THAT CAN BE CENTERED AT ANY POINT OF THE PLANE AND THAT, WHEN OPERATED, REMOVES FROM THE PLANE PRECISELY THOSE POINTS WHOSE DISTANCE FROM THE CENTER IS IRRATIONAL. HOW MANY PUNCHES ARE NEEDED TO REMOVE EVERY POINT?

FIRST NOTE THAT TWO PUNCHES ARE NOT ENOUGH. INDEED, SUPPOSE WE PUNCH AT TWO POINTS P AND Q. CHOOSE SOME RATIONAL NUMBER r GREATER THAN ONE HALF THE DISTANCE FROM P TO Q AND CONSIDER THE CIRCLES OF RADIUS r ABOUT P AND Q.



THESE CIRCLES INTERSECT AND ANY POINT R OF INTERSECTION IS A RATIONAL DISTANCE FROM BOTH P AND Q AND SO IS NOT REMOVED BY EITHER PUNCH.

HOWEVER, WE CLAIM THAT ONE MORE (CAREFULLY CHOSEN) PUNCH WILL REMOVE EVERYTHING. WE CLAIM THAT, AFTER THE TWO PUNCHES AT P AND Q ONLY COUNTABLY MANY POINTS REMAIN. INDEED, EACH PUNCH LEAVES BEHIND COUNTABLY MANY CIRCLES AND TO BE LEFT BEHIND AFTER BOTH PUNCHES A POINT MUST BE ON THE INTERSECTION OF TWO OF THESE. BUT TWO NONCONCENTRIC CIRCLES CAN

INTERSECT IN AT MOST TWO POINTS SO THE TOTAL NUMBER OF THESE IS COUNTABLE.

ABOUT EACH ONE OF THESE REMAINING POINTS CONSIDER ALL OF THE CIRCLES OF RATIONAL RADIUS (A COUNTABLE NUMBER OF CIRCLES ALTOGETHER).

FIX SOME STRAIGHT LINE L AND LET S BE ITS SET OF INTERSECTION POINTS WITH THESE CIRCLES. THEN S IS COUNTABLE.

SELECT SOME P_0 IN $L - S$. THEN P_0 IS AN IRRATIONAL DISTANCE FROM ALL OF THE POINTS LEFT BEHIND AFTER THE FIRST TWO PUNCHES SO ONE MORE PUNCH AT P_0 WILL REMOVE THEM ALL.