# SUMMATION OF SERIES

RECALL THAT AN INFINITE SERIES $\sum_{n=0}^{\infty} a_n$ CONVERGES TO $S$ IF AND ONLY IF ITS SEQUENCE $\{a_0, a_0+a_1, a_0+a_1+a_2, \ldots\}$ OF PARTIAL SUMS CONVERGES TO $S$. GENERALLY, ONE SHOWS THAT A SERIES CONVERGES (TO SOMETHING) BY APPLYING SOME SORT OF "TEST" THAT GIVES LITTLE OR NO INFORMATION ABOUT THE ACTUAL SUM $S$ OF THE SERIES, E.G., IT IS EASY TO SHOW THAT $\sum_{n=1}^{\infty} \frac{1}{n^2}$ CONVERGES (SAY, BY THE INTEGRAL TEST), BUT THE SUM IS FAR FROM OBVIOUS (IN FACT, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, BUT THE USUAL WAY TO SHOW THIS USES FOURIER SERIES EXPANSIONS). WORSE YET, IT IS JUST AS EASY TO SHOW THAT $\sum_{n=1}^{\infty} \frac{1}{n^3}$ CONVERGES, BUT TO THIS DAY NO ONE KNOWS THE EXACT VALUE OF THE SUM (ALAN BAKER RECEIVED THE FIELDS MEDAL FOR DEVELOPING TECHNIQUES THAT MANAGED TO SHOW ONLY THAT IT IS IRRATIONAL).

WE WILL ILLUSTRATE JUST A FEW OF THE MORE ELEMENTARY TECHNIQUES FOR SUMMING SERIES. ALL OF THESE ARE INTRODUCED IN EVERY CALCULUS COURSE, BUT APPLYING THEM IS GENERALLY VERY TRICKY. THEY ARE BASED ON

1. <u>THE GEOMETRIC SERIES</u> : $\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}$ FOR $|x| < 1$

<u>SIMPLE EXAMPLE</u> :

$$\begin{aligned}\sum_{n=1}^{\infty} 3^n 5^{1-n} &= \sum_{n=1}^{\infty} 5\left(\tfrac{3}{5}\right)^n \\ &= \sum_{n=0}^{\infty} 5\left(\tfrac{3}{5}\right)^n - 5 \\ &= \frac{5}{1-\frac{3}{5}} - 5 \\ &= \frac{15}{2}\end{aligned}$$

2. TELESCOPING SERIES :

SIMPLE EXAMPLE : $\sum_{n=1}^{\infty} \ln(1+\frac{1}{n}) = \sum_{n=1}^{\infty} \ln(\frac{n+1}{n}) = \sum_{n=1}^{\infty} (\ln(n+1) - \ln n)$

SO THE $k^{TH}$ PARTIAL SUM IS

$$S_k = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + (\ln 4 - \ln 3) + \cdots$$
$$(\ln k - \ln(k-1)) + (\ln(k+1) - \ln k)$$
$$= -\ln 1 + \ln(k+1)$$
$$= \ln(k+1)$$

AND $\lim_{k\to\infty} S_k = \lim_{k\to\infty} \ln(k+1)$ WHICH DOES NOT EXIST SO $\sum_{n=1}^{\infty} \ln(1+\frac{1}{n})$ DIVERGES.

3. POWER SERIES EXPANSIONS :

SIMPLE EXAMPLE : TO SUM THE SERIES $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$ (WHICH CONVERGES BY THE ALTERNATING SERIES TEST) WE BEGIN WITH THE SERIES EXPANSION

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \quad , \quad |x| < 1$$

(GOTTEN BY INTEGRATING $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ ). THE SERIES ACTUALLY CONVERGES ALSO AT $x = 1$ (A.S.T.). THAT ITS SUM IS $\ln 2$ FOLLOWS FROM ABEL'S THEOREM : IF $\sum a_n x^n$ CONVERGES AT $x = R > 0$, THEN $\sum a_n R^n = \lim_{x\to R^-} \sum a_n x^n$. THUS,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \lim_{x\to 1^-} \ln(1+x) = \ln 2 .$$

## EXAMPLES :

1. SUM THE SERIES

$$\frac{1^2}{0!} + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \cdots$$

THE SERIES IS $\sum_{n=0}^{\infty} \frac{(n+1)^2}{n!}$ SO WE BEGIN WITH

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \qquad , \; -\infty < x < \infty$$

THEN

$$x e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^{n+1}$$

SO

$$(xe^x)' = \sum_{n=0}^{\infty} \frac{n+1}{n!} x^n$$

$$xe^x + e^x = \sum_{n=0}^{\infty} \frac{n+1}{n!} x^n$$

$$x^2e^x + xe^x = \sum_{n=0}^{\infty} \frac{n+1}{n!} x^{n+1}$$

$$(x^2e^x + xe^x)' = \sum_{n=0}^{\infty} \frac{(n+1)^2}{n!} x^n$$

$$x^2e^x + 2xe^x + xe^x + e^x = \sum_{n=0}^{\infty} \frac{(n+1)^2}{n!} x^n$$

AT $x = 1$ THIS GIVES

$$5e = \sum_{n=0}^{\infty} \frac{(n+1)^2}{n!}$$

2. SUM THE SERIES

$$\sum_{n=1}^{\infty} 3^{n-1} \sin^3\left(\frac{x}{3^n}\right)$$

FACTS WORTH REMEMBERING :

EULER'S FORMULA : $e^{\theta i} = \cos\theta + i\sin\theta$

DE MOIVRE'S FORMULA : $(e^{\theta i})^n = e^{n\theta i} = \cos n\theta + i\sin n\theta$

WE USE DE MOIVRE'S FORMULA AS FOLLOWS :

$$(e^{\theta i})^3 = (\cos\theta + i\sin\theta)^3$$

$$\cos 3\theta + i\sin 3\theta = (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

SO

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

$$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$$

$$= 3\sin\theta - 4\sin^3\theta$$

THUS,

$$\sin^3\theta = \tfrac{3}{4}\sin\theta - \tfrac{1}{4}\sin 3\theta$$

SO

$$3^{n-1}\sin^3\left(\tfrac{x}{3^n}\right) = \tfrac{1}{4}\left[3^n\sin\left(\tfrac{x}{3^n}\right) - 3^{n-1}\sin\left(\tfrac{x}{3^{n-1}}\right)\right]$$

NOW,

$$\sum_{n=1}^{\infty} 3^{n-1}\sin^3\left(\tfrac{x}{3^n}\right) = \sum_{n=1}^{\infty} \tfrac{1}{4}\left[3^n\sin\left(\tfrac{x}{3^n}\right) - 3^{n-1}\sin\left(\tfrac{x}{3^{n-1}}\right)\right]$$

LOOKS LIKE IT SHOULD TELESCOPE. THE $k^{TH}$ PARTIAL SUM IS

$$S_k = \tfrac{1}{4}\Big[\left(3\sin\left(\tfrac{x}{3}\right) - \sin x\right) + \left(3^2\sin\left(\tfrac{x}{3^2}\right) - 3\sin\left(\tfrac{x}{3}\right)\right) +$$

$$\left(3^3\sin\left(\tfrac{x}{3^3}\right) - 3^2\sin\left(\tfrac{x}{3^2}\right)\right) + \left(3^4\sin\left(\tfrac{x}{3^4}\right) - 3^3\sin\left(\tfrac{x}{3^3}\right)\right) + \cdots$$

$$+ \left(3^k\sin\left(\tfrac{x}{3^k}\right) - 3^{k-1}\sin\left(\tfrac{x}{3^{k-1}}\right)\right)\Big]$$

$$= \tfrac{1}{4}\left[-\sin x + 3^k\sin\left(\tfrac{x}{3^k}\right)\right]$$

THUS,

$$\lim_{k\to\infty} S_k = -\tfrac{1}{4}\sin x + \tfrac{1}{4}\lim_{k\to\infty} 3^k \sin\left(\frac{x}{3^k}\right)$$

$$= -\tfrac{1}{4}\sin x + \tfrac{1}{4}\lim_{k\to\infty} x \underbrace{\left(\frac{\sin\left(\frac{x}{3^k}\right)}{\frac{x}{3^k}}\right)}_{\to 1}$$

$$= -\tfrac{1}{4}\sin x + \tfrac{1}{4}x$$

$$= \sum_{n=1}^{\infty} 3^{n-1}\sin^3\left(\frac{x}{3^n}\right)$$

3. SUM THE SERIES

$$\sum_{(p,q)=1} \frac{1}{x^{p+q}-1}$$

FOR $|x| > 1$, WHERE THE SUM IS OVER ALL POSITIVE INTEGERS $p$ AND $q$ THAT ARE RELATIVELY PRIME.

FIRST NOTE THAT

$$\sum_{(p,q)=1} \frac{1}{x^{p+q}-1} = \sum_{(p,q)=1} \frac{1}{x^{p+q}}\left(\frac{1}{1-\frac{1}{x^{p+q}}}\right) = \sum_{(p,q)=1} \frac{1}{x^{p+q}}\left(\sum_{n=0}^{\infty}\left(\frac{1}{x^{p+q}}\right)^n\right)$$

BECAUSE $|x| > 1$

$$= \sum_{(p,q)=1}\sum_{n=0}^{\infty}\left(\frac{1}{x^{p+q}}\right)^{n+1}$$

$$= \sum_{(p,q)=1}\sum_{n=0}^{\infty}\frac{1}{x^{(p+q)(n+1)}}$$

$$= \sum_{(p,q)=1}\sum_{n=1}^{\infty}\frac{1}{x^{np+nq}} = \sum_{(p,q)=1}\sum_{n=1}^{\infty}\frac{1}{x^{np}}\frac{1}{x^{nq}}$$

NOW, FOR EACH CHOICE OF $p, q$ AND $n$ WITH $(p,q)=1$ AND $n \geq 1$ SET

$$i = np$$
$$j = nq$$

NOTE THAT ANY $i, j = 1, 2, \ldots$ CAN BE WRITTEN IN THIS FORM FOR SUCH A CHOICE OF $p, q$ AND $n$ AND THAT, MOREOVER, THIS CAN BE DONE IN EXACTLY ONE WAY:

GIVEN $i$ AND $j$, LET $n = (i, j)$. THEN $\left(\frac{i}{n}, \frac{j}{n}\right) = 1$

SO IF WE LET $p = \frac{i}{n}$ AND $q = \frac{j}{n}$ WE HAVE

$$i = np$$
$$j = nq$$
$$n \geq 1 \quad \text{AND} \quad (p,q) = 1$$

TO PROVE UNIQUENESS, SUPPOSE

$$i = n'p'$$
$$j = n'q'$$
$$n' \geq 1 \quad \text{AND} \quad (p',q') = 1$$

THEN $(p',q') = 1 \Rightarrow n' = (i,j) = n \Rightarrow p' = \frac{i}{n} = p$

AND $q' = \frac{j}{n} = q$.

THUS, THE SERIES CAN BE WRITTEN

$$\sum_{(p,q)=1} \frac{1}{x^{p+q}-1} = \sum_{(p,q)=1} \sum_{n=1}^{\infty} \frac{1}{x^{np}} \frac{1}{x^{nq}} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{x^i} \frac{1}{x^j} = \sum_{i=1}^{\infty} \frac{1}{x^i} \left( \sum_{j=1}^{\infty} \frac{1}{x^j} \right)$$

$$= \left( \sum_{i=1}^{\infty} \frac{1}{x^i} \right) \left( \sum_{j=1}^{\infty} \frac{1}{x^j} \right) = \left( \sum_{i=1}^{\infty} \frac{1}{x^i} \right)^2 = \left( \sum_{i=0}^{\infty} \left(\frac{1}{x}\right)^i - 1 \right)^2$$

$$= \left( \frac{1}{1-\frac{1}{x}} - 1 \right)^2 = \frac{1}{(x-1)^2}.$$