PROBLEM SOLVING SEMINAR

SYLLABUS PROBLEMS (SOLUTIONS)

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1. SUPPOSE THE PLANE IS COLORED WITH TWO COLORS; SOME POINTS ARE RED AND SOME POINTS ARE BLUE. MUST THERE BE TWO POINTS AN INCH APART THAT HAVE THE SAME COLOR?

THE ANSWER IS "YES". DRAW ANY EQUILATERAL TRIANGLE OF SIDE LENGTH 1. THERE ARE 3 VERTICES AND EACH MUST BE ONE OF TWO COLORS SO AT LEAST 2 OF THEM MUST HAVE THE SAME COLOR (THIS IS AN INSTANCE OF THE "PIGEONHOLE PRINCIPLE" WHICH WE WILL DISCUSS MORE FULLY A BIT LATER).

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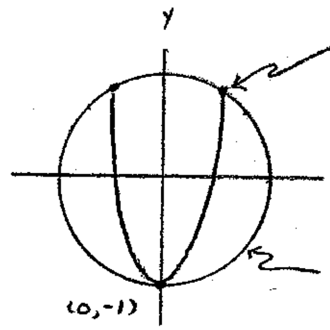
2. CAN AN ARC OF A PARABOLA INSIDE A CIRCLE OF RADIUS 1 HAVE LENGTH GREATER THAN 4?

THE ANSWER IS "YES". TO SEE THIS WE CONSIDER PARABOLAS

$$y = ax^2 - 1$$

FOR a SUFFICIENTLY LARGE.



$x^2+(ax^2-1)^2=1 \Rightarrow$

$x=0, \pm \frac{1}{a}\sqrt{2a-1}$

(ASSUMING $a > \frac{1}{2}$)

ARC LENGTH OF THE PARABOLA FROM $X=0$ TO $X=\frac{1}{a}\sqrt{2a-1}$ IS

$$\int_0^{\frac{1}{a}\sqrt{2a-1}} \sqrt{1+4a^2x^2}\,dx = \frac{1}{2a}\int_0^{2\sqrt{2a-1}} \sqrt{1+u^2}\,du$$

EVALUATING THE INTEGRAL DIRECTLY GIVES

$$\frac{1}{2a}\left[\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln(u+\sqrt{1+u^2})\right]_0^{2\sqrt{2a-1}}$$

WHICH DOESN'T HELP MUCH (IS THERE AN a FOR WHICH THIS IS > 2 ?).

HERE IS A TRICK FOR ESTIMATING THE INTEGRAL.

$$\frac{1}{2a}\int_0^{2\sqrt{2a-1}} \sqrt{1+u^2}\,du = \frac{1}{2a}\int_0^{2\sqrt{2a-1}} [\sqrt{1+u^2} - u + u]\,du =$$

$$\frac{1}{2a}\int_0^{2\sqrt{2a-1}} [\sqrt{1+u^2} - u]\,du + \frac{1}{2a}\left[\frac{1}{2}u^2\Big|_0^{2\sqrt{2a-1}}\right] =$$

$$\frac{1}{2a}\int_0^{2\sqrt{2a-1}} [\sqrt{1+u^2} - u]\,du + 2 - \frac{1}{a}$$

NOW WE NEED ONLY SHOW THAT FOR SOME a, $\int_0^{2\sqrt{2a-1}} [\sqrt{1+u^2}-u]\,du$ EXCEEDS 2. BUT, BY RATIONALIZING,

$$\sqrt{1+u^2} - u = \frac{1}{\sqrt{1+u^2}+u} > \frac{1}{\sqrt{1+u^2}+\sqrt{1+u^2}} = \frac{1}{2\sqrt{1+u^2}}$$

$$> \frac{1}{2(\sqrt{1}+\sqrt{u^2})} = \frac{1}{2(1+u)}$$

AND SINCE $\int_0^\infty \frac{1}{2(1+u)}du$ DIVERGES, SO DOES $\int_0^\infty [\sqrt{1+u^2}-u]du$.

THUS, FOR SUFFICIENTLY LARGE a,

$$\int_0^{2\sqrt{2a-1}} [\sqrt{1+u^2}-u]du > 2.$$

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3. SHOW THAT IF $\sum_{n=1}^{\infty} a_n$ IS A CONVERGENT SERIES OF POSITIVE REAL NUMBERS, THEN SO IS $\sum_{n=1}^{\infty} (a_n)^{\frac{n}{n+1}}$.

NOTE THAT

$$(a_n)^{\frac{n}{n+1}} = (a_n)^{1-\frac{1}{n+1}} = \frac{a_n}{(a_n)^{\frac{1}{n+1}}}$$

IF $a_n \geq \frac{1}{2^{n+1}}$, THEN $(a_n)^{\frac{1}{n+1}} \geq \frac{1}{2}$ SO

$$(a_n)^{\frac{n}{n+1}} \leq 2a_n .$$

ON THE OTHER HAND, IF $a_n \leq \frac{1}{2^{n+1}}$, THEN

$(a_n)^{\frac{n}{n+1}} \leq \left(\frac{1}{2^{n+1}}\right)^{\frac{n}{n+1}} = \frac{1}{2^n}$. THUS, IN EITHER CASE

$$(a_n)^{\frac{n}{n+1}} \leq 2a_n + \frac{1}{2^n} .$$

SINCE $\sum_{n=1}^{\infty} (2a_n + \frac{1}{2^n})$ CONVERGES AND THE $(a_n)^{\frac{n}{n+1}} \geq 0$, THE SERIES $\sum_{n=1}^{\infty} (a_n)^{\frac{n}{n+1}}$ CONVERGES BY THE COMPARISON TEST.

4. ALICE AND BOB PLAY A GAME IN WHICH THEY TAKE TURNS REMOVING SOME STONES FROM A HEAP THAT INITIALLY HAS n STONES. THE NUMBER OF STONES REMOVED AT EACH TURN MUST BE ONE LESS THAN A PRIME NUMBER. THE WINNER IS THE PLAYER WHO TAKES THE LAST STONE. ALICE PLAYS FIRST. SHOW THAT THERE ARE INFINITELY MANY n FOR WHICH BOB HAS A WINNING STRATEGY.

TO SAY THAT BOB HAS A WINNING STRATEGY FOR SOME n MEANS THAT, HOWEVER ALICE MOVES INITIALLY, BOB HAS A SEQUENCE OF MOVES GUARANTEED TO WIN (E.G., IF $n = 3$ ALICE CAN TAKE EITHER 2 OR 1 LEAVING 1 OR 2 AND BOB CAN TAKE THE REMAINING STONES TO WIN).

NOW WE SUPPOSE TO THE CONTRARY THAT BOB HAS A WINNING STRATEGY ONLY FOR FINITELY MANY INITIAL HEAP SIZES, SAY, $B = \{b_1, \ldots, b_m\}$. NOTICE THAT IF ALICE HAS A WINNING STRATEGY FOR SOME n, THEN n MUST BE OF THE FORM

$$p-1$$

OR

$$b + (p-1)$$

FOR SOME PRIME p AND SOME $b \in B$. THE REASON IS AS FOLLOWS:

CERTAINLY, ALICE HAS A WINNING STRATEGY IF $n = p-1$ (" TAKE THEM ALL"). IF n IS NOT OF THIS FORM, BUT ALICE HAS A WINNING STRATEGY SHE MUST HAVE A MOVE THAT RESULTS IN A HEAP SIZE FOR WHICH THE SECOND TO PLAY (WHICH SHE WILL BE AFTER HER FIRST MOVE) HAS A WINNING STRATEGY, I.E., THERE MUST BE A PRIME p FOR WHICH $n-(p-1) \in B$ SO $n = b + (p-1)$ FOR SOME $b \in B$.

NEXT NOTICE THAT FOR ANY n IN $\mathbb{N}$, ONE OF ALICE OR BOB MUST HAVE A WINNING STRATEGY. WE PROVE THIS BY INDUCTION. CHECK OUT THE FIRST FEW VALUES OF n.

$n = 1$	ALICE WINS	(" TAKE 1")
$n = 2$	ALICE WINS	(" TAKE 2")
$n = 3$	BOB WINS	(SEE THE STRATEGY ABOVE)
$n = 4$	ALICE WINS	(" TAKE 4")
$n = 5$	ALICE WINS	(ALICE TAKES 2, LEAVING 3 SO THE SECOND TO PLAY AFTER THIS (ALICE) WINS)

NOW SUPPOSE THAT FOR ALL $k < n$ ONE OF ALICE OR BOB HAS A WINNING STRATEGY FOR A HEAP OF SIZE k. CONSIDER A HEAP OF SIZE n. THERE ARE TWO POSSIBILITIES :

1. THERE EXISTS SOME PRIME p FOR WHICH $k-(p-1)$ IS IN B. IN THIS CASE, ALICE WINS. (" TAKE $p-1$"; THEN ALICE PLAYS SECOND SO $k-(p-1)$ IN B IMPLIES ALICE WINS.)

2. FOR EVERY PRIME p, $k-(p-1)$ IS NOT IN B. THEN BOB WINS (WHATEVER ALICE DOES, THE SECOND TO PLAY (NOW ALICE) DOES NOT HAVE A WINNING STRATEGY SO, BY THE INDUCTION HYPOTHESIS, THE FIRST TO PLAY (NOW BOB) DOES).

THE CONCLUSION IS THAT FOR ANY n NOT IN B, ALICE HAS A WINNING STRATEGY, I.E., n MUST EITHER BE $p-1$ OR $b+(p-1)$ FOR SOME PRIME p AND SOME b IN B. WE SHOW NOW THAT THIS IS IMPOSSIBLE IF B IS FINITE.

LET t BE ANY INTEGER LARGER THAN ALL OF THE b IN B. THEN

$$(t+1)!+2,\ (t+1)!+3,\ \ldots,\ (t+1)!+(t+1)$$

IS A SEQUENCE OF t COMPOSITE INTEGERS. TAKE

$$n = (t+1)! + t$$

THEN n IS NOT $p-1$ FOR ANY PRIME p (INDEED, IT IS $[(t+1)!+(t+1)]-1$). MOREOVER, FOR ANY b IN B, $n \neq b+(p-1)$ FOR ANY PRIME p BECAUSE $n-b+1$ IS ONE OF THE COMPOSITES LISTED ABOVE.

5. SUPPOSE I IS A HALF-OPEN INTERVAL IN $\mathbb{R}$ AND $f: I \to I$ IS A CONTINUOUS FUNCTION SATISFYING THE FOLLOWING CONDITION: FOR EACH x IN I THERE IS A (LEAST) POSITIVE INTEGER $N(x)$ SUCH THAT $f^{N(x)}(x) = x$ [HERE $f^0(x) = x$, $f^1(x) = f(x)$, $f^2(x) = f(f(x)), \ldots, f^n(x) = f(f^{n-1}(x)), \ldots$]. SHOW THAT f MUST BE THE IDENTITY FUNCTION.

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f MAPS I ONTO I: $\quad x \in I \Rightarrow x = f(f^{N(x)-1}(x))$

f IS ONE-TO-ONE: $\quad f(x) = f(y) \Rightarrow$

$$x = f^{N(x)}(x) = f^{N(x)}(f^{N(x)}(x)) = f^{2N(x)}(x) = \cdots = f^{N(y)N(x)}(x)$$

AND, SIMILARLY,

$$y = f^{N(y)}(y) = f^{N(x)N(y)}(y) = f^{N(y)N(x)}(y)$$

BUT

$$\begin{aligned} f^{N(y)N(x)}(x) &= f^{N(y)N(x)-1}(f(x)) \\ &= f^{N(y)N(x)-1}(f(y)) \\ &= f^{N(y)N(x)}(y) \end{aligned}$$

SO $x = y$.

BEING A CONTINUOUS BIJECTION OF I ONTO I, f IS STRICTLY MONOTONE, I.E., EITHER STRICTLY INCREASING OR STRICTLY DECREASING (IF THIS IS NOT CLEAR TO YOU THERE IS A PROOF AT THE END OF THIS SOLUTION).

WE FIRST SHOW THAT IF f IS INCREASING, THEN IT MUST BE THE IDENTITY FUNCTION. INDEED, SUPPOSE f IS INCREASING, BUT

$f(x) \neq x$ FOR SOME $x \in I$. THEN

$$x < f(x) \Rightarrow x < f(x) < f^2(x) < \cdots < f^{N(x)}(x) = x$$

WHICH IS IMPOSSIBLE, AND

$$x > f(x) \Rightarrow x > f(x) > f^2(x) > \cdots > f^{N(x)}(x) = x$$

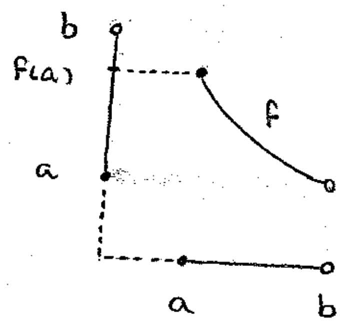
WHICH IS ALSO IMPOSSIBLE

THUS, $f(x) = x \quad \forall x \in I$.

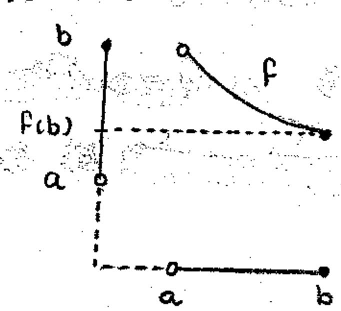
<u>NOTE</u>: NONE OF THIS SO FAR DEPENDS ON I BEING A HALF-OPEN INTERVAL.

FINALLY, WE SHOW THAT f CANNOT BE DECREASING:

IF $I = [a, b)$, THEN $f(a) < b$ SO NO ELEMENT OF THE INTERVAL $(f(a), b)$ CAN BE IN THE IMAGE OF f



AND THIS CONTRADICTS THE SURJECTIVITY OF f. IF $I = (a, b]$, THEN $f(b) > a$ SO NO ELEMENT OF THE INTERVAL $(a, f(b))$ CAN BE IN THE IMAGE OF f.



FINALLY, HERE IS THE LITTLE LEMMA WE USED IN THE PROOF.

<u>LEMMA</u> : LET $I \subseteq \mathbb{R}$ BE AN INTERVAL AND SUPPOSE $f : I \to I$ IS A CONTINUOUS BIJECTION. THEN f IS EITHER STRICTLY INCREASING OR STRICTLY DECREASING.

<u>PROOF</u> : SUPPOSE NOT. THEN WE CAN FIND $a < b < c$ IN I FOR WHICH EITHER

$$\text{(i)} \quad f(a) < f(b) \quad \text{AND} \quad f(b) > f(c)$$

OR

$$\text{(ii)} \quad f(a) > f(b) \quad \text{AND} \quad f(b) < f(c)$$

(BECAUSE f IS A BIJECTION). SUPPOSE WE ARE IN CASE (i). SINCE f IS CONTINUOUS THE INTERMEDIATE VALUE THEOREM IMPLIES THAT THE IMAGE OF $[a,b]$ IS $[f(a), f(b)]$ AND THE IMAGE OF $[b,c]$ IS $[f(c), f(b)]$. BUT THEN

$$[f(a), f(b)] \cap [f(c), f(b)]$$

IS A NONEMPTY INTERVAL AND ANY y IN THE INTERIOR OF THIS INTERVAL IS THE IMAGE OF TWO POINTS IN I (ONE FROM (a,b) AND ONE FROM (b,c)) AND THIS CONTRADICTS THE FACT THAT f IS ONE-TO-ONE. CASE (ii) IS EXACTLY THE SAME. $\square$

FINALLY, NOTICE THAT THE RESULT IS FALSE IF I IS EITHER OPEN OR CLOSED. FOR EXAMPLE, IF I IS EITHER $(0,1)$ OR $[0,1]$ THE FUNCTION $f(x) = -x+1$ IS CONTINUOUS AND SATISFIES $f^2(x) = f(f(x)) = -f(x)+1 = -(-x+1)+1 = x$ FOR EVERY x.