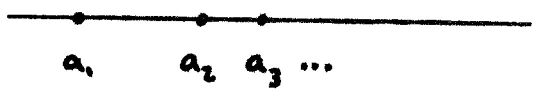
MONOTONE SEQUENCES

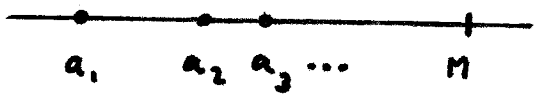
FOR MANY SEQUENCES OF THE TYPE WE WILL BE MOST INTERESTED IN SOON IT IS IMPORTANT TO KNOW <u>THAT</u> THEY CONVERGE, EVEN IF IT IS VERY DIFFICULT TO KNOW <u>WHAT</u> THEY CONVERGE TO.

INTUITIVELY, THE IDEA IS THIS :

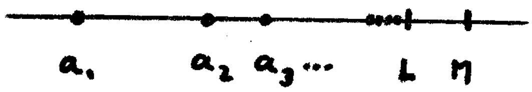
IF THE TERMS OF A SEQUENCE $\{a_1, a_2, a_3, \ldots\}$ ARE MOVING TO THE RIGHT



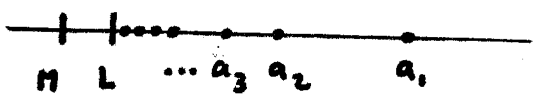
ON THE NUMBER LINE, BUT CAN'T GET PAST THE REAL NUMBER M



THEN THEY HAVE TO "BUNCH UP" SOMEWHERE BEFORE M



AND SO HAVE A LIMIT THERE. OF COURSE, YOU CAN TURN THE PICTURE AROUND ALSO.



TO SAY ALL OF THIS PRECISELY WE NEED SOME DEFINITIONS:

A SEQUENCE $\{a_n\}_{n=1}^{\infty}$ IS

<u>STRICTLY</u> <u>INCREASING</u> IF $a_1 < a_2 < a_3 < \cdots$

<u>INCREASING</u> IF $a_1 \leq a_2 \leq a_3 \leq \cdots$

<u>STRICTLY</u> <u>DECREASING</u> IF $a_1 > a_2 > a_3 > \cdots$

<u>DECREASING</u> IF $a_1 \geq a_2 \geq a_3 \geq \cdots$

IT'S <u>MONOTONE</u> IF IT IS EITHER INCREASING OR DECREASING AND <u>STRICTLY</u> <u>MONOTONE</u> IF IT IS EITHER STRICTLY INCREASING OR STRICTLY DECREASING.

E.G., $\{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ IS STRICTLY DECREASING,

$\{1, 2, 2, 3, 3, 3, \ldots\}$ IS INCREASING AND

$\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots\}$ IS NEITHER INCREASING NOR DECREASING.

OF COURSE, A STRICTLY INCREASING SEQUENCE IS ALSO INCREASING (AND SIMILARLY FOR DECREASING).

IT IS NOT ALWAYS OBVIOUS THAT A SEQUENCE HAS ONE OF THESE PROPERTIES, BUT HERE ARE A FEW WAYS OF DECIDING :

1. COMPUTE $a_{n+1} - a_n$

EXAMPLE : $\left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty} = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \right\}$

$$a_n = \frac{n}{n+1}$$

$$a_{n+1} = \frac{n+1}{(n+1)+1} = \frac{n+1}{n+2}$$

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1}$$

$$= \frac{n+1}{n+2}\,\frac{n+1}{n+1} - \frac{n}{n+1}\,\frac{n+2}{n+2}$$

$$= \frac{(n+1)(n+1) - n(n+2)}{(n+1)(n+2)} = \frac{n^2+2n+1-n^2-2n}{(n+1)(n+2)}$$

$$= \frac{1}{(n+1)(n+2)}$$ WHICH IS ALWAYS POSITIVE

$$a_{n+1} - a_n > 0$$

$$a_{n+1} > a_n$$ SO THE SEQUENCE IS (STRICTLY) INCREASING

2. COMPUTE $\frac{a_{n+1}}{a_n}$

EXAMPLE : $\left\{ \frac{n}{3n+2} \right\}_{n=1}^{\infty}$

$$a_n = \frac{n}{3n+2}$$

$$a_{n+1} = \frac{n+1}{3(n+1)+2} = \frac{n+1}{3n+5}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{3n+5}}{\frac{n}{3n+2}} = \frac{n+1}{3n+5} \cdot \frac{3n+2}{n} = \frac{3n^2+5n+2}{3n^2+5n} > 1$$

$\frac{a_{n+1}}{a_n} > 1$ AND $a_n > 0$ SO

$$a_{n+1} > a_n$$

AND THE SEQUENCE IS (STRICTLY) INCREASING.

3. COMPUTE THE DERIVATIVE

<u>EXAMPLE</u> : $\left\{ \frac{3n}{2n+1} \right\}_{n \geq 1}^{\infty}$

$a_n = \frac{3n}{2n+1}$

<u>NOTE</u> : THIS ISN'T A DIFFERENTIABLE FUNCTION, BUT $\frac{3x}{2x+1}$ IS AND WE JUST WON'T BOTHER TO CHANGE THE ns TO xs.

$$a_n' = \left(\frac{3n}{2n+1} \right)' = \frac{(2n+1)(3) - (3n)(2)}{(2n+1)^2} = \frac{3}{(2n+1)^2} > 0$$

SO $a_n = \frac{3n}{2n+1}$ IS (STRICTLY) INCREASING.

<u>ANOTHER</u> <u>EXAMPLE</u> : $\left\{ \dfrac{10^n}{n!} \right\}_{n=1}^{\infty}$

$$a_n = \frac{10^n}{n!}$$

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{10^{n+1}}{(n+1)!}}{\dfrac{10^n}{n!}} = \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n} = \frac{10}{n+1}$$

NOTICE THAT

$$\frac{10}{n+1} \quad \text{IS} \quad \begin{cases} > 1 & \text{IF } n = 1, 2, \ldots, 8 \\ = 1 & \text{IF } n = 9 \\ < 1 & \text{IF } n \geq 10 \end{cases}$$

THUS, THE SEQUENCE INCREASES FOR A LITTLE WHILE

$$a_1 < a_2 < a_3 < a_4 < a_5 < a_6 < a_7 < a_8 < a_9$$

REPEATS ONCE

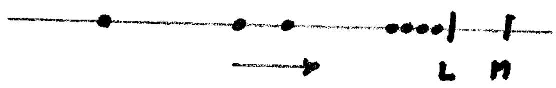
$$a_9 = a_{10}$$

BUT IS <u>EVENTUALLY</u> (STRICTLY) DECREASING

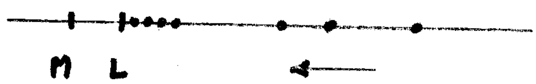
$$a_{10} > a_{11} > a_{12} > a_{13} > \cdots$$

AS FAR AS CONVERGENCE AND DIVERGENCE ARE CONCERNED, ONLY THE "EVENTUAL" BEHAVIOR OF A SEQUENCE MATTERS.

IF A SEQUENCE $\{a_n\}$ IS EVENTUALLY INCREASING AND BOUNDED FROM ABOVE ($a_n \leq M$), THEN IT HAS A LIMIT $L \leq M$.



IF A SEQUENCE $\{a_n\}$ IS EVENTUALLY DECREASING AND BOUNDED FROM BELOW ($M \leq a_n$), THEN IT HAS A LIMIT $L \geq M$.



<u>EXAMPLE</u> : $\left\{ \frac{10^n}{n!} \right\}_{n=1}^{\infty}$ IS EVENTUALLY DECREASING (LAST EXAMPLE) AND CLEARLY BOUNDED FROM BELOW BY $M = 0$

$$0 \leq \frac{10^n}{n!}$$

SO IT HAS SOME LIMIT $L \geq 0$.

NOTICE THAT THE RESULTS DO NOT TELL US WHAT

$$L = \lim_{n\to\infty} \frac{10^n}{n!}$$

ACTUALLY IS, BUT ONLY THAT IT EXISTS AND IS ≥ 0.

USUALLY, FINDING THE LIMIT REQUIRES MUCH MORE WORK.

IN THIS CASE, IT ISN'T TOO BAD:

$$a_{n+1} = \frac{10^{n+1}}{(n+1)!} = \frac{10}{n+1}\,\frac{10^n}{n!} = \frac{10}{n+1}\,a_n$$

NOW TAKE THE LIMIT AS $n\to\infty$ ON BOTH SIDES OF

$$a_{n+1} = \frac{10}{n+1}\,a_n$$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(\frac{10}{n+1}\,a_n\right)$$

$$L = 0\cdot L$$

$$L = 0$$

SO

$$\lim_{n\to\infty} \frac{10^n}{n!} = 0 .$$

SIMILARLY,

$$\lim_{n\to\infty} \frac{c^n}{n!} = 0$$

FOR ANY CONSTANT C.