MATH 680 LECTURE 14 MORE ON LIE GROUPS

SOME THINGS WE KNOW ALREADY :

1. $GL(n,\mathbb{R})$ = INVERTIBLE $n \times n$ REAL MATRICES
IS AN OPEN SUBMANIFOLD OF $\mathbb{R}^{n^2}$ AND SO IS A LIE GROUP.
$T_{id}(GL(n,\mathbb{R}))$ = ALL OF $\mathbb{R}^{n^2}$, WHICH IS CERTAINLY CLOSED UNDER FORMATION OF COMMUTATORS. WITH THIS BRACKET OPERATION, $T_{id}(GL(n,\mathbb{R}))$ IS CALLED THE <u>LIE ALGEBRA OF</u> $GL(n,\mathbb{R})$ AND WRITTEN

$$\mathfrak{gl}(n,\mathbb{R}).$$

2. EVERYTHING IS THE SAME FOR $GL(n,\mathbb{C})$. ITS LIE ALGEBRA (ALL OF $\mathbb{R}^{2n^2}$) IS WRITTEN

$$\mathfrak{gl}(n,\mathbb{C}).$$

3. $O(n)$ IS A SUBMANIFOLD OF $\mathbb{R}^{n^2}$ AND $SO(n)$ IS AN OPEN SUBMANIFOLD OF $O(n)$ AND THEY ARE BOTH LIE GROUPS. WE'LL FIND THEIR LIE ALGEBRAS IN A MOMENT.

4. $SO(3)$ IS A LIE GROUP WITH LIE ALGEBRA

$$\mathfrak{so}(3) = \text{ALL SKEW-SYMMETRIC } 3\times 3 \text{ REAL MATRICES.}$$

IN THIS CASE,

$$\exp : \mathfrak{so}(3) \longrightarrow SO(3)$$

IS A SURJECTION.

<u>EXERCISE</u>: SHOW THAT

$$\exp : \mathfrak{gl}(n, \mathbb{R}) \to GL(n, \mathbb{R})$$

IS A SMOOTH MAP WHOSE DERIVATIVE $\exp_{*0}$ AT $0 \in \mathfrak{gl}(n, \mathbb{R})$ IS THE IDENTITY MAP. SHOW, HOWEVER, THAT $\exp$ DOES <u>NOT</u> MAP ONTO $GL(n, \mathbb{R})$ IN THIS CASE.

TO FIND $T_{id}(O(n))$ WE CONSIDER A SMOOTH CURVE $g(t)$ IN $O(n)$ WITH $g(0) = id$ AND NOTE THAT

$$g(t)\,g(t)^T = id$$

FOR EVERY t. DIFFERENTIATING WITH RESPECT TO t AT $t = 0$ GIVES

$$g(0)\,g'(0)^T + g'(0)\,g(0)^T = 0$$

$$(id)\,g'(0)^T + g'(0)\,(id)^T = 0$$

$$g'(0)^T + g'(0) = 0$$

$$g'(0)^T = -g'(0)$$

SO EVERY ELEMENT OF $T_{id}(O(n))$ IS SKEW-SYMMETRIC. ON THE OTHER HAND, IF A IS ANY $n \times n$ REAL SKEW-SYMMETRIC MATRIX, THEN $A^T = -A$ SO

$$\exp(A)^T = \exp(A^T) = \exp(-A) = \exp(A)^{-1}$$

SO $\exp(A) \in O(n)$. MOREOVER, FOR ANY $t \in \mathbb{R}$,

tA IS ALSO SKEW-SYMMETRIC SO

$$t \to \exp(tA)$$

IS A SMOOTH CURVE IN $O(n)$ THAT GOES THROUGH id AT $t=0$. IT'S VELOCITY VECTOR AT $t=0$ IS A SO $A \in T_{id}(O(n))$, I.E.,

$T_{id}(O(n))$ = ALL SKEW-SYMMETRIC $n \times n$ REAL MATRICES

WE ALREADY KNOW THAT THIS IS CLOSED UNDER COMMUTATOR SO, WITH THIS BRACKET OPERATION, $T_{id}(O(n))$ IS A LIE ALGEBRA, CALLED THE <u>LIE ALGEBRA OF</u> $O(n)$ AND DENOTED

$$\mathfrak{o}(n).$$

SINCE $SO(n)$ IS AN OPEN SUBMANIFOLD OF $O(n)$, IT HAS THE SAME TANGENT SPACE AT id.

$\mathfrak{so}(n) = \mathfrak{o}(n)$ = ALL $n \times n$ REAL SKEW-SYMMETRIC MATRICES

NEXT WE WOULD LIKE TO DO THE SAME SORT OF THING FOR

$SL(n, \mathbb{R})$ = ALL $n \times n$ REAL MATRICES g WITH $\det g = 1$

$SL(n, \mathbb{C})$ = ALL $n \times n$ COMPLEX MATRICES g WITH $\det g = 1$

$U(n)$ = ALL $n \times n$ COMPLEX MATRICES

g WITH $g\bar{g}^T = \bar{g}^T g = id$

$SU(n)$ = ALL $g \in U(n)$ WITH $\det g = 1$

UNFORTUNATELY, WE HAVE NOT YET SHOWN THAT THESE ARE EVEN SMOOTH MANIFOLDS IN GENERAL (EXCEPT $SU(2)$ WHICH IS JUST S^3).

<u>EXERCISE</u> : SHOW DIRECTLY THAT $SL(2,\mathbb{R})$ IS A SUBMANIFOLD OF $\mathbb{R}^4$ OF DIMENSION 3 AND SO, IN PARTICULAR, IS A LIE GROUP. SHOW THAT $T_{id}(SL(2,\mathbb{R}))$ CONSISTS PRECISELY OF THE 2×2 REAL MATRICES A WITH $Tr(A) = 0$ AND THAT THIS IS CLOSED UNDER CONJUGATION.

$\mathfrak{sl}(2,\mathbb{R})$ = THE LIE ALGEBRA OF 2×2 REAL TRACEFREE MATRICES

FOR THE REMAINING $SL(n,\mathbb{R})$ AS WELL AS $U(n)$ AND $SU(n)$ WE TAKE A DIFFERENT APPROACH.

CONSIDER THE MAP

$$\exp : \mathfrak{gl}(n,\mathbb{R}) \to GL(n,\mathbb{R})$$

AND SUPPOSE $A \in \mathfrak{gl}(n,\mathbb{R})$ IS TRACEFREE ($Tr(A) = 0$).

THEN

$$\det(\exp(A)) = e^{Tr(A)} = e^0 = 1$$

SO

$$\exp(A) \in SL(n, \mathbb{R})$$

THE SET OF TRACEFREE ELEMENTS OF $\mathfrak{gl}(n, \mathbb{R})$ IS A LINEAR SUBSPACE THAT IS CLOSED UNDER COMMUTATOR AND WHICH exp CARRIES INTO $SL(n, \mathbb{R})$.

IT WOULD BE NICE IF exp CARRIED THE TRACEFREE ELEMENTS OF $\mathfrak{gl}(n, \mathbb{R})$ ONTO $SL(n, \mathbb{R})$, BUT, UNFORTUNATELY, THIS IS NOT TRUE. IN FACT, EVEN SUCH A SIMPLE THING AS

$$g = \begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \in SL(2, \mathbb{R})$$

CANNOT BE WRITTEN AS $\exp(A)$ FOR ANY 2×2 REAL TRACEFREE MATRIX A.

<u>NOTE</u>: THE REASON IS THAT ONE CAN SHOW THAT FOR ANY 2×2 REAL TRACEFREE MATRIX A, $Tr(\exp(A))$ CANNOT BE LESS THAN -2, BUT $Tr(g) = -\frac{5}{2}$.

HOWEVER, WE SHOW NOW THAT <u>LOCALLY</u> exp CARRIES THE TRACEFREE MATRICES IN $gl(n,\mathbb{R})$ ONTO $SL(n,\mathbb{R})$:

LET U BE AN OPEN NEIGHBORHOOD OF 0 IN $gl(n,\mathbb{R})$ ON WHICH exp IS A DIFFEOMORPHISM ONTO THE OPEN NEIGHBORHOOD $\exp(U)$ OF id IN $GL(n,\mathbb{R})$.

$\exp(U) \cap SL(n,\mathbb{R})$ IS AN OPEN NEIGHBORHOOD OF id IN $SL(n,\mathbb{R})$.

GIVEN $g \in \exp(U) \cap SL(n,\mathbb{R})$ $\exists!$ $A \in U$ SUCH THAT

$$\exp(A) = g$$

MOREOVER, A MUST BE TRACEFREE SINCE

$$g \in SL(n,\mathbb{R}) \Rightarrow 1 = \det g = \det(\exp(A)) = e^{Tr(A)}$$

$$\Rightarrow Tr(A) = 0$$

<u>CONCLUSION</u>: exp CARRIES $U \cap$ (TRACEFREE MATRICES) DIFFEOMORPHICALLY ONTO THE OPEN NEIGHBORHOOD $\exp(U) \cap SL(n,\mathbb{R})$ OF id IN $SL(n,\mathbb{R})$.

THUS,

$$\exp^{-1} \big|_{\exp(U) \cap SL(n,\mathbb{R})}$$

IS A CHART AT id IN $SL(n,\mathbb{R})$.

SINCE THE TRACEFREE MATRICES IN $\mathfrak{gl}(n,\mathbb{R}) = \mathbb{R}^{n^2}$ FORM A HYPERPLANE $(A_{11} + A_{22} + \cdots + A_{nn} = 0)$ THIS IS A SUBMANIFOLD CHART FROM $GL(n,\mathbb{R})$. THE DIMENSION IS

$$n^2 - 1 .$$

<u>EXERCISE</u>: SHOW THAT BY COMPOSING WITH LEFT (OR RIGHT) TRANSLATIONS OF $SL(n,\mathbb{R})$ ONE CAN PRODUCE SUBMANIFOLD CHARTS AT ANY $g \in SL(n,\mathbb{R})$.

THUS, WE CONCLUDE THAT $SL(n,\mathbb{R})$ IS A SMOOTH SUBMANIFOLD OF $GL(n,\mathbb{R})$ OF DIMENSION n^2-1 AND THEREFORE A LIE GROUP. ITS LIE ALGEBRA IS

$$\mathfrak{sl}(n,\mathbb{R}) = \{ A \in \mathfrak{gl}(n,\mathbb{R}) : Tr(A) = 0 \}$$

THE SAME ARGUMENTS WORK IN THE COMPLEX CASE TO SHOW THAT $SL(n,\mathbb{C})$ IS A LIE GROUP WITH LIE ALGEBRA

$$\mathfrak{sl}(n,\mathbb{C}) = \{ A \in \mathfrak{gl}(n,\mathbb{C}) : Tr(A) = 0 \}.$$

NOTING THAT, FOR $A \in \mathfrak{gl}(n,\mathbb{C})$,

A SKEW-HERMITIAN ($\bar{A}^T = -A$) $\Rightarrow$

$$\overline{\exp(A)}^T = \exp(\bar{A}^T) = \exp(-A) = \exp(A)^{-1}$$

$\Rightarrow \quad \exp(A) \in U(n)$

AND

A SKEW-HERMITIAN AND TRACEFREE $\Longrightarrow$

$$\exp(A) \in SU(n)$$

ONE CAN ARGUE BASICALLY AS IN THE CASE OF $SL(n, \mathbb{R})$ TO SHOW THAT

$U(n)$ IS A LIE GROUP WITH LIE ALGEBRA

$$\mathfrak{u}(n) = \{ A \in \mathfrak{gl}(n, \mathbb{C}) : \bar{A}^T = -A \}$$

AND

$SU(n)$ IS A LIE GROUP WITH LIE ALGEBRA

$$\mathfrak{su}(n) = \{ A \in \mathfrak{gl}(n, \mathbb{C}) : \bar{A}^T = -A \text{ AND } Tr(A) = 0 \}$$

THE DIMENSIONS OF THESE (AS REAL VECTOR SPACES) ARE n^2 AND $n^2 - 1$, RESPECTIVELY.

__EXERCISE__ (SO(3) AND SU(2)): SO(3) AND SU(2) ARE __NOT__ THE SAME LIE GROUP (THE FIRST IS DIFFEOMORPHIC TO $\mathbb{RP}^3$ AND THE SECOND TO S^3). HERE YOU WILL SHOW THAT, NEVERTHELESS, THEY HAVE " THE SAME " LIE ALGEBRA.

__DEFINITION__: TWO LIE ALGEBRAS $(V_1, [\ ,\ ]_1)$ AND $(V_2, [\ ,\ ]_2)$ ARE __ISOMORPHIC__ IF THERE IS A VECTOR SPACE ISOMORPHISM $T : V_1 \to V_2$ OF V_1 ONTO V_2 SUCH THAT

$$[v, w]_1 = [Tv, Tw]_2$$

FOR ALL $v, w \in V_1$.

WE KNOW THAT

$\mathfrak{so}(3)$ = ALL 3×3 REAL MATRICES THAT ARE SKEW-SYMMETRIC

AND WE HAVE ALREADY FOUND A BASIS FOR IT:

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

1. SHOW THAT, FOR $i, j, k = 1, 2, 3$,

$$[L_i, L_j] = \sum_{k=1}^{3} \varepsilon_{ijk} L_k$$

WHERE ε_{ijk} IS THE "LEVI-CIVITA SYMBOL" (1 IF ijk IS AN

EVEN PERMUTATION OF 123, -1 IF ijk IS AN ODD PERMUTATION OF 123 AND 0 OTHERWISE).

WE ALSO KNOW THAT

$$su(2) = \text{ALL } 2\times 2 \text{ COMPLEX MATRICES } A \text{ WITH } \bar{A}^T = -A \text{ AND } Tr(A) = 0$$

2. SHOW THAT, IF

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

ARE THE PAULI SPIN MATRICES, THEN

$$S_1 = -\frac{i}{2}\sigma_1 \quad , \quad S_2 = -\frac{i}{2}\sigma_2 \quad , \quad S_3 = -\frac{i}{2}\sigma_3$$

FORM A BASIS FOR $su(2)$.

3. SHOW THAT, FOR $i, j, k = 1, 2, 3$,

$$[S_i, S_j] = \sum_{k=1}^{3} \varepsilon_{ijk} S_k$$

4. USE # 1 AND # 3 TO SHOW THAT $so(3)$ AND $su(2)$ ARE ISOMORPHIC LIE ALGEBRAS.