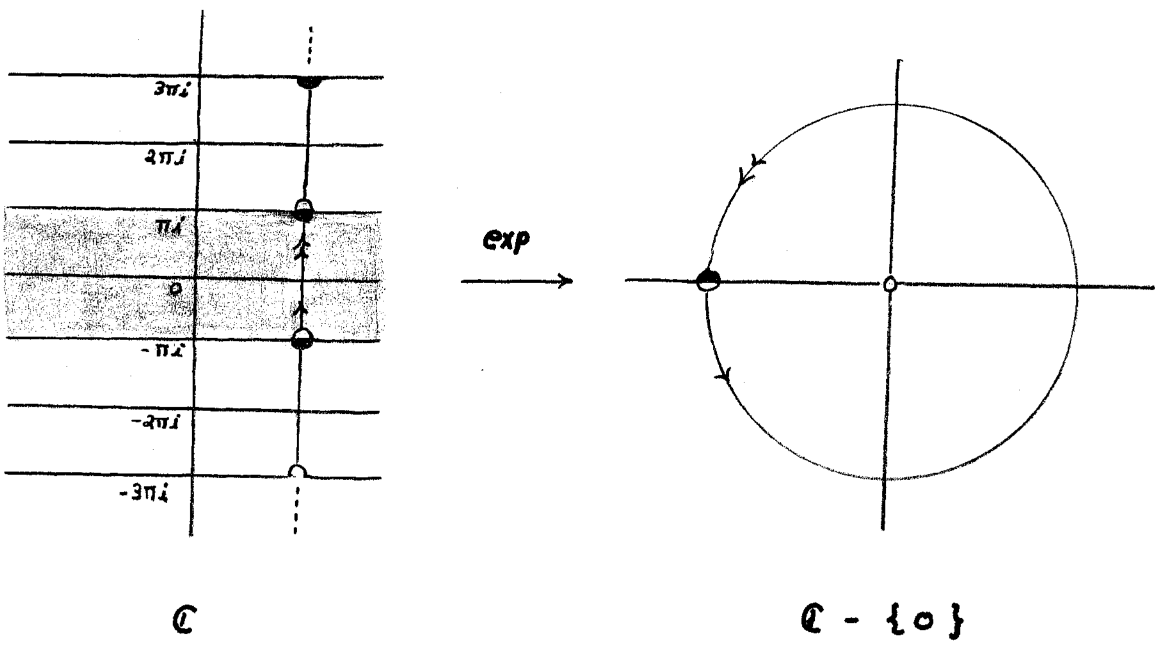
"MULTI-VALUED FUNCTIONS" AND BRANCHES :

LET'S BE CLEAR ABOUT THIS : THERE IS NO SUCH THING AS A "MULTI-VALUED FUNCTION". A FUNCTION TAKES EXACTLY ONE VALUE AT EACH POINT IN ITS DOMAIN. THE TERMINOLOGY IS OLD AND OBSOLETE, BUT STILL AROUND.

THE COMPLEX ANALOGUES OF CERTAIN FAMILIAR FUNCTIONS (E.G., $\sqrt{x}$, $\ln x$, ...) ARE MORE SUBTLE THAN THOSE WE HAVE SEEN SO FAR. ALL OF THESE CAN BE DEFINED IN TERMS OF THE LOGARITHM (E.G., $\sqrt{x} = e^{\frac{1}{2}\ln x}$ FOR $x > 0$) SO WE CONSIDER THIS ONE FIRST.

log IS SUPPOSED TO BE THE INVERSE OF exp. THE PROBLEM IS THAT THE COMPLEX EXPONENTIAL MAP DOES NOT HAVE AN INVERSE.



EXP MAPS ALL OF

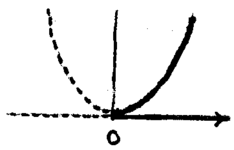
$$\ln\rho + (\phi + 2k\pi)i \quad , \; k = 0, \pm 1, \pm 2, \ldots$$

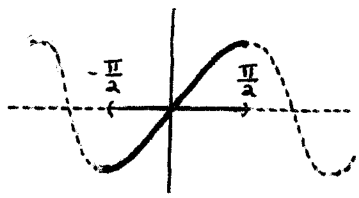
ONTO THE SAME VALUE

$$\rho e^{i\phi}$$

AND AN INVERSE <u>FUNCTION</u> IS ONLY ALLOWED TO PICK ONE.

THE SAME PROBLEM OCCURS WHEN TRYING TO DEFINE INVERSES FOR FUNCTIONS LIKE x^2 AND $\sin x$. THE SOLUTION IS TO RESTRICT THE DOMAIN OF THE FUNCTION.

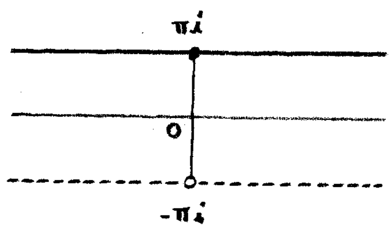




INVERSE : $\sqrt{x}$ INVERSE : ARCSIN X

OTHER CHOICES ARE ALSO POSSIBLE

RESTRICTING EXP TO ITS FUNDAMENTAL REGION



GIVES A MAP ONTO $\mathbb{C} - \{0\}$ WITH AN INVERSE, CALLED THE <u>PRINCIPAL BRANCH OF THE LOGARITHM</u> AND WRITTEN

$$\log z = \ln |z| + i \arg z$$

E.G.,

$$\log(1-i) = \log(\sqrt{2}\, e^{-\frac{\pi}{4}i}) = \ln\sqrt{2} + i(-\tfrac{\pi}{4}) = \tfrac{1}{2}(\ln 2 - \tfrac{\pi}{2} i)$$

OTHER <u>BRANCHES OF THE LOGARITHM</u> ARE OBTAINED BY RESTRICTING exp TO OTHER HALF-OPEN HORIZONTAL STRIPS OF WIDTH 2π (THERE ARE NO SPECIAL NAMES OR SYMBOLS FOR THESE). EACH IS DEFINED ON ALL OF $\mathbb{C} - \{0\}$ AND MAPS INTO ITS HORIZONTAL STRIP.

IT IS TRADITIONAL TO LUMP ALL OF THESE BRANCHES TOGETHER INTO ONE OBJECT CALLED THE "MULTI-VALUED" <u>COMPLEX LOGARITHM FUNCTION</u> (ALTHOUGH IT IS NOT A FUNCTION AT ALL) AND DENOTED

$$\log z = \ln|z| + i(\arg z + 2k\pi) \qquad k = 0, \pm 1, \pm 2, \ldots$$

<u>NOTE</u>: IT IS POSSIBLE TO TURN $\log z$ INTO AN HONEST FUNCTION BY TAKING ITS DOMAIN TO BE NOT $\mathbb{C} - \{0\}$, BUT AN OBJECT KNOWN AS A <u>RIEMANN SURFACE</u>. WE WILL NOT GET INTO THIS, HOWEVER.

A FEW SIMPLE PROPERTIES :

1. $z = re^{i\theta} \Rightarrow \log z = \log(re^{i\theta}) = \ln r + (\theta + 2k\pi)i$

$k = 0, \pm 1, \pm 2, \ldots$

PROOF : $r = |z|$ AND $\theta = \arg z + 2k'\pi$ FOR SOME k'

2. $e^{\log z} = z$

PROOF : $e^{\log z} = e^{(\ln r + (\theta + 2k\pi)i)} = e^{\ln r} e^{(\theta + 2k\pi)i}$

$= r e^{\theta i + 2k\pi i}$

$= re^{i\theta} = z$

3. $\log(e^z) \neq z$ BECAUSE THE RIGHT-HAND SIDE IS A SINGLE NUMBER, BUT THE LEFT-HAND SIDE IS A SET OF NUMBERS

IN FACT,

$$\log(e^z) = z + 2k\pi i \quad , \; k = 0, \pm 1, \pm 2, \ldots$$

PROOF : $\log(e^z) = \log(e^{x+yi}) = \log(e^x e^{yi})$

$= \ln e^x + (y + 2k\pi)i$

$= (x + yi) + 2k\pi i = z + 2k\pi i$

4. $\log(e^z) \neq z$ UNLESS $-\pi < \mathrm{Im}(z) \leq \pi$

5. $\log(z_1 z_2) \neq \log z_1 + \log z_2$ IN GENERAL

E.G., $z_1 = z_2 = -1 = e^{\pi i} \Rightarrow \log z_1 + \log z_2 = \log(e^{\pi i}) + \log(e^{\pi i})$

$= \pi i + \pi i = 2\pi i$

BUT $\log(z_1 z_2) = \log(e^{2\pi i}) = \log(e^{0i}) = 0$

6. $\log(z_1 z_2) = \log z_1 + \log z_2$ IN THE SENSE THAT A VALUE OF $\log z_1$ PLUS A VALUE OF $\log z_2$ IS SOME VALUE OF $\log(z_1 z_2)$

PROOF :

A VALUE OF $\log z_1$: $\ln r_1 + (\theta_1 + 2k_1\pi)i$

A VALUE OF $\log z_2$: $\ln r_2 + (\theta_2 + 2k_2\pi)i$

SUM : $(\ln r_1 + \ln r_2) + \left((\theta_1+\theta_2) + 2(k_1+k_2)\pi\right)i$

$= \ln(r_1 r_2) + \left((\theta_1+\theta_2) + 2k\pi\right)i$

$(k = k_1 + k_2)$

WHICH IS A VALUE OF $\log\left((r_1 r_2)e^{i(\theta_1+\theta_2)}\right) = \log(z_1 z_2)$.

SIMILIARLY,

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2 \qquad (z_2 \neq 0)$$

EXERCISES :

27. COMPUTE

(a) $\text{Log}(-ei)$ ANS. $1 - \frac{\pi}{2}i$

(b) $\log e$ ANS. $1 + 2k\pi i$

(c) $\log i$ ANS. $(2k + \frac{1}{2})\pi i$

(d) $\text{Log}(-1+\sqrt{3}i)$ ANS. $\ln 2 + \frac{2\pi}{3}i$

28. SHOW THAT $\log(i^2) \neq 2\log i$, BUT $\text{Log}(i^2) = 2\,\text{Log}\,i$

29. SHOW THAT IF $Re(z_1) > 0$ AND $Re(z_2) > 0$, THEN
$\log(z_1 z_2) = \log z_1 + \log z_2$.

30. FIND ALL SOLUTIONS TO THE EQUATION

$$\log z = \frac{\pi}{2} i$$

(MEANING THAT SOME LOGARITHM OF z IS $\frac{\pi}{2} i$).

31. PROVE THAT IF n IS A NON-NEGATIVE INTEGER AND $z \neq 0$, THEN

$$z^n = e^{n \log z}$$

(IN PARTICULAR, THE RIGHT-HAND SIDE CAN TAKE ON ONLY ONE VALUE).

32. PROVE THAT IF n IS A NON-NEGATIVE INTEGER AND $z \neq 0$, THEN

$$z^{\frac{1}{n}} = e^{\frac{1}{n} \log z}$$

(IN PARTICULAR, THE RIGHT-HAND SIDES TAKES ON EXACTLY n VALUES)

THE LAST TWO EXERCISES MOTIVATE THE FOLLOWING DEFINITION.

LET c BE AN ARBITRARY COMPLEX NUMBER. FOR $z \neq 0$, DEFINE

$$z^c = e^{c \log z}$$

THIS IS, IN GENERAL, "MULTI-VALUED" AND ITS PRINCIPAL VALUE IS DEFINED TO BE

$$e^{c \operatorname{Log} z}$$

E.G.,

$$(-i)^{i} = e^{i\log(-i)} = e^{i\log(e^{-\frac{\pi}{2}i})}$$

$$= e^{i(\ln 1 + (-\frac{\pi}{2}+2k\pi)i)}$$

$$= e^{i(-\frac{\pi}{2}+2k\pi)i} = e^{\frac{\pi}{2}-2k\pi}$$

ALL OF THESE ARE REAL NUMBERS AND THE PRINCIPAL VALUE IS $e^{\frac{\pi}{2}}$.

SIMILARLY, IF C IS ANY NONZERO COMPLEX NUMBER AND Z IS ARBITRARY WE DEFINE

$$C^{z} = e^{z\log C}$$

NOTE : C IS A CONSTANT AND ONCE A VALUE OF $\log C$ IS FIXED THIS BECOMES A SINGLE-VALUED FUNCTION, E.G., WHEN $C = e$ AND THE PRINCIPAL VALUE IS SELECTED ONE OBTAINS THE "USUAL" FUNCTION e^{z} $(\log e = 1)$.

INVERSE TRIGONOMETRIC AND INVERSE HYPERBOLIC FUNCTIONS ARE ALSO DEFINED IN TERMS OF THE LOGARITHM. FOR EXAMPLE, WE SOLVE

$$z = \sin w$$

FOR w AS FOLLOWS :

$$z = \frac{1}{2i}(e^{iw} - e^{-iw})$$

$$[\; z = \frac{1}{2i}(e^{iw} - e^{-iw})\;]\; 2ie^{iw}$$

$$2iz e^{iw} = (e^{iw})^2 - 1$$

$$(e^{iw})^2 - 2iz(e^{iw}) - 1 = 0 \qquad \text{(QUADRATIC IN } e^{iw}\text{)}$$

$$e^{iw} = \frac{2iz + \sqrt{4-4z^2}}{2} \qquad (\sqrt{} \text{ MEANS EITHER VALUE})$$

$$= iz + \sqrt{1-z^2}$$

$$iw = \log(iz + \sqrt{1-z^2})$$

$$w = -i \log(iz + \sqrt{1-z^2})$$

THUS, WE DEFINE

$$\text{ARCSIN } z = -i \log(iz + \sqrt{1-z^2})$$

<u>NOTE</u> : THE "MULTI-VALUEDNESS" THIS TIME COMES FROM TWO SOURCES : $\sqrt{}$ AND $\log$

E.G.,

$$\text{ARCSIN } i = -i \log(-1 + \sqrt{2}) = \begin{cases} -i \log(-1+\sqrt{2}) \\ -i \log(-1-\sqrt{2}) \end{cases}$$

HERE $\sqrt{2}$ MEANS THE USUAL POSITIVE SQUARE ROOT OF 2.

$$= \begin{cases} -i \log((-1+\sqrt{2})e^{0i}) \\ -i \log((1+\sqrt{2})e^{\pi i}) \end{cases}$$

$$= \begin{cases} -i(\ln(-1+\sqrt{2}) + 2k\pi i) \\ -i(\ln(1+\sqrt{2}) + (\pi + 2k\pi)i \end{cases}$$

$$= \begin{cases} 2k\pi - i\ln(-1+\sqrt{2}) \\ (2k+1)\pi - i\ln(1+\sqrt{2}) \end{cases}$$

<u>NOTE</u> : $-\ln(-1+\sqrt{2}) = \ln\left(\frac{1}{\sqrt{2}-1}\right) = \ln\left(\frac{1}{\sqrt{2}-1}\,\frac{\sqrt{2}+1}{\sqrt{2}+1}\right)$

$$= \ln\left(\frac{\sqrt{2}+1}{2-1}\right)$$

$$= \ln(1+\sqrt{2})$$

THUS,

$$\text{ARCSIN } i = \begin{cases} 2k\pi + i\ln(1+\sqrt{2}) \\ (2k+1)\pi - i\ln(1+\sqrt{2}) \end{cases}$$

$$= \begin{cases} 2k\pi + (-1)^{2k}\ln(1+\sqrt{2})\,i \\ (2k+1)\pi + (-1)^{2k+1}\ln(1+\sqrt{2})\,i \end{cases}$$

$$= n\pi + (-1)^n \ln(1+\sqrt{2})\,i$$

$$n = 0, \pm 1, \pm 2, \cdots$$

<u>EXERCISE</u> 33 : SOLVE $z = \tan w$ FOR w TO ARRIVE AT THE DEFINITION

$$\text{ARCTAN } z = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right)$$

AND THEN COMPUTE

$$\text{ARCTAN }(2i).$$

ANS. $-\frac{(2n+1)\pi}{2} + \frac{\ln 3}{2}\,i$

$n = 0, \pm 1, \pm 2, \cdots$

INVERSE HYPERBOLIC FUNCTIONS ARE TREATED SIMILARLY, E.G.,

$z = \sinh w$ LEADS TO

$$\sinh^{-1} z = \log (z + \sqrt{z^2+1})$$

EXERCISES :

34. COMPUTE $(1+i)^i$ ANS. $e^{-(\frac{\pi}{4}+2k\pi)} e^{(\ln\sqrt{2})i}$

35. FIND THE PRINCIPAL VALUE OF $\left[\frac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i}$

ANS. $-e^{2\pi^2}$

36. SHOW THAT IF $z \neq 0$ AND a IS REAL, THEN

$$|z^a| = |z|^a$$

37. LET C BE A NONZERO COMPLEX NUMBER. WHAT CAN BE SAID ABOUT C IF $|i^c|$ IS CONSTANT ?

38. FIND ALL SOLUTIONS TO THE EQUATION

$$\sin z = 2.$$

ANS. $(2k+\frac{1}{2})\pi \pm i \ln(2+\sqrt{3})$

SOLUTIONS TO THE EXERCISES :

27. (a) $\mathcal{Log}(-ei) = \mathcal{Log}(e\,e^{-\frac{\pi}{2}i}) = \ln e + i(-\frac{\pi}{2}) = 1 - \frac{\pi}{2}i$

(b) $\log e = \log(e\,e^{0i}) = \ln e + (0+2k\pi)i = 1 + 2k\pi i$

(c) $\log i = \log(1e^{\frac{\pi}{2}i}) = \ln 1 + (\frac{\pi}{2}+2k\pi)i = (2k+\frac{1}{2})\pi i$

(d) $\mathcal{Log}(-1+\sqrt{3}i) = \mathcal{Log}(2e^{\frac{2\pi}{3}i}) = \ln 2 + \frac{2\pi}{3}i$

28. $\log(i^2) = \log(-1) = \log(1\cdot e^{\pi i}) = \ln 1 + (\pi + 2k\pi)i$

$= (2k+1)\pi i$

$\mathcal{Log}(i^2) = \pi i$

$2\log i = 2\log(1\cdot e^{\frac{\pi}{2}i}) = 2(\ln 1 + (\frac{\pi}{2}+2k\pi)i)$

$= (4k+1)\pi i \neq \log(i^2)$

$2\,\mathcal{Log}\, i = \pi i = \mathcal{Log}(i^2)$

29. $z_1 = r_1 e^{i\theta_1}$ $\quad : \mathrm{Re}(z_1) > 0 \Rightarrow -\frac{\pi}{2} < \theta_1 < \frac{\pi}{2}$

$z_2 = r_2 e^{i\theta_2}$ $\quad : \mathrm{Re}(z_2) > 0 \Rightarrow -\frac{\pi}{2} < \theta_2 < \frac{\pi}{2}$

$\Rightarrow -\pi < \theta_1 + \theta_2 < \pi$

$\Rightarrow \mathcal{Log}(z_1 z_2) = \mathcal{Log}(r_1 r_2 e^{i(\theta_1+\theta_2)})$

$= \ln(r_1 r_2) + (\theta_1 + \theta_2)i$

$= (\ln r_1 + \theta_1 i) + (\ln r_2 + \theta_2 i)$

$= \mathcal{Log}\, z_1 + \mathcal{Log}\, z_2$

30. $\log z = \frac{\pi}{2} i \Rightarrow e^{\log z} = e^{\frac{\pi}{2} i}$

$$z = i$$

31. $z = re^{i\theta} \Rightarrow e^{n \log z} = e^{n[\ln r + i(\theta + 2k\pi)]}$

$$= e^{n \ln r + (n\theta + 2(nk)\pi)i}$$

$$= e^{n \ln r} e^{(n\theta i + 2(nk)\pi i)}$$

$$= r^n e^{n\theta i} \quad \text{BECAUSE } nk \text{ IS AN INTEGER}$$

$$= (re^{\theta i})^n = z^n$$

32. $z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{(\frac{\theta + 2k\pi}{n})i}, \quad k = 0, \ldots, n-1$

$$e^{\frac{1}{n} \log z} = e^{\frac{1}{n}(\ln r + (\theta + 2k\pi)i)}, \quad k = 0, \pm 1, \pm 2, \ldots$$

$$= e^{\frac{1}{n} \ln r} e^{(\frac{\theta + 2k\pi}{n})i}, \quad k = 0, \pm 1, \pm 2, \ldots$$

$$= r^{\frac{1}{n}} e^{(\frac{\theta + 2k\pi}{n})i}, \quad k = 0, \pm 1, \pm 2, \ldots$$

$$= r^{\frac{1}{n}} e^{(\frac{\theta + 2k\pi}{n})i}, \quad k = 0, \ldots, n-1$$

BECAUSE exp IS PERIODIC WITH PERIOD $2\pi i$

33. $z = \tan w = \frac{\sin w}{\cos w} = \frac{\frac{1}{2i}(e^{iw} - e^{-iw})}{\frac{1}{2}(e^{iw} + e^{-iw})}$

$$z = \frac{1}{i} \frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} \frac{e^{iw}}{e^{iw}}$$

$$z = \frac{1}{i} \frac{e^{2wi} - 1}{e^{2wi} + 1}$$

$$zi(e^{2wi} + 1) = e^{2wi} - 1$$

33. (CONTINUED)

$$(2i-1)e^{2wi} = -2i-1$$

$$e^{2wi} = \frac{-2i-1}{2i-1} = \frac{1+2i}{1-2i}\,\frac{i}{i} = \frac{i-2}{i+2}$$

$$2wi = \log\left(\frac{i-z}{i+z}\right)$$

$$w = -\frac{i}{2}\log\left(\frac{i-z}{i+z}\right)$$

$$= \frac{i}{2}\log\left(\frac{i+z}{i-z}\right)$$

THUS,

$$\text{ARCTAN}\,(2i) = \frac{i}{2}\log\left(\frac{i+2i}{i-2i}\right)$$

$$= \frac{i}{2}\log\left(\frac{3i}{-i}\right) = \frac{i}{2}\log(-3)$$

$$= \frac{i}{2}\log\left(3e^{\pi i}\right)$$

$$= \frac{i}{2}\left(\ln 3 + (\pi+2k\pi)i\right)$$

$$= -\frac{(2k+1)\pi}{2} + \frac{\ln 3}{2}i \quad , k = 0, \pm 1, \pm 2, \ldots$$

34. $$(1+i)^i = e^{i\log(1+i)} = e^{i\log(\sqrt{2}e^{\frac{\pi}{4}i})}$$

$$= e^{i[\ln\sqrt{2} + (\frac{\pi}{4}+2k\pi)i]}$$

$$= e^{-(\frac{\pi}{4}+2k\pi)}\,e^{(\ln\sqrt{2})i}$$

35. $$\left[\frac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i} = e^{3\pi i\,[\log(\frac{e}{2}(-1-\sqrt{3}i))]}$$

$$= e^{3\pi i\,[\log(ee^{-\frac{2\pi}{3}i})]} = e^{3\pi i\,[\ln e - \frac{2\pi}{3}i]}$$

$$= e^{2\pi^2+3\pi i} = e^{2\pi^2}e^{3\pi i} = -e^{2\pi^2}$$

36. $z \neq 0$ AND a REAL.

$$z^a = e^{a \log z} = e^{a(\ln r + i(\theta + 2k\pi))}$$
$$= e^{a \ln r + i(a\theta + 2k\pi a)}$$
$$= e^{a \ln r} e^{i(a\theta + 2k\pi a)}$$

$|z^a| = e^{a \ln r}$ BECAUSE a IS REAL

$$= r^a$$
$$= |z|^a$$

37. SUPPOSE $|i^c|$ IS CONSTANT

WRITE $c = a + bi$. THEN

$$i^c = e^{c \log i}$$
$$= e^{c \log(e^{\frac{\pi}{2} i})} = e^{c((\frac{\pi}{2} + 2k\pi)i)}$$
$$= e^{(a+bi)((\frac{\pi}{2}+2k\pi)i)} = e^{-b(\frac{\pi}{2}+2k\pi)} e^{a(\frac{\pi}{2}+2k\pi)i}$$

SO
$$|i^c| = e^{-b(\frac{\pi}{2}+2k\pi)}$$

AND THIS CAN BE CONSTANT (INDEPENDENT OF k) ONLY IF $b = 0$, I.E., IF c IS REAL.

38. $\sin z = 2 \Rightarrow$

$$z = \arcsin 2 = -i \log(i(2) + \sqrt{1-2^2})$$
$$= -i \log(2i \pm \sqrt{3}\, i) = -i \log((2 \pm \sqrt{3}) i)$$
$$= -i \log((2 \pm \sqrt{3}) e^{\frac{\pi}{2} i}) = -i\left[\ln(2 \pm \sqrt{3}) + (\tfrac{\pi}{2} + 2k\pi) i\right]$$
$$= (2k + \tfrac{1}{2})\pi - \ln(2 \pm \sqrt{3}) i$$
$$= (2k + \tfrac{1}{2})\pi \pm \ln(2 + \sqrt{3}) i \qquad \text{BECAUSE } 2 - \sqrt{3} = \frac{1}{2+\sqrt{3}}$$