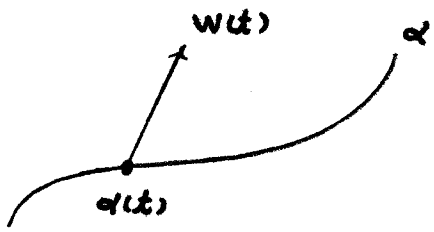
M A MANIFOLD WITH CONNECTION ∇, V AND W VECTOR FIELDS ON M. THEN THE VECTOR FIELD $\nabla_V W$ HAS A VALUE AT $p \in M$ THAT DEPENDS ONLY ON

1. THE VALUE OF V AT p,
2. THE VALUES OF W IN SOME (ARBITRARY) NEIGHBORHOOD OF p.

FOR MANY APPLICATIONS (PARALLEL TRANSLATION, GEODESICS, ETC.) NEED TO DIFFERENTIATE "VECTOR FIELDS ALONG CURVES"



IN THE DIRECTION OF THE VELOCITY VECTOR $\alpha'(t)$.

<ins>DEFINITION</ins>: LET M BE A SMOOTH MANIFOLD AND (TM, π, M) ITS TANGENT BUNDLE. SUPPOSE

$$\alpha : I \to M$$

IS A SMOOTH CURVE IN M. A <ins>VECTOR FIELD ALONG</ins> α IS A SMOOTH

MAP

$$X : I \to TM$$

SUCH THAT

$$\pi(X(t)) = \alpha(t)$$

$$\begin{tikzcd} & TM \arrow[d, "\pi"] \\ I \arrow[ur, "X"] \arrow[r, "\alpha"'] & M \end{tikzcd}$$

FOR EACH $t \in I$.

THE SET OF ALL SUCH VECTOR FIELDS ALONG α IS DENOTED

$$\mathcal{T}_\alpha(TM)$$

AND GIVEN THE OBVIOUS POINTWISE ALGEBRAIC STRUCTURE :

$$(X+Y)(t) = X(t) + Y(t)$$

$$(fX)(t) = f(t)X(t) \qquad \text{FOR } f \in C^\infty(I)$$

<u>EXAMPLE</u> : THE TANGENT FIELD TO α

$$t \longrightarrow (\alpha(t), \alpha'(t))$$

NOW WE SUPPOSE M HAS A CONNECTION ∇. WE SHOW THAT ∇ GIVES A UNIQUE "REASONABLE" WAY OF DIFFERENTIATING $X \in \mathcal{T}_\alpha(TM)$ ALONG α.

<u>THEOREM</u>: LET M BE A SMOOTH MANIFOLD WITH A CONNECTION ∇. THEN, FOR ANY SMOOTH CURVE $\alpha : I \to M$, THERE IS A UNIQUE OPERATOR

$$\frac{D}{dt} : T_\alpha(TM) \longrightarrow T_\alpha(TM)$$

(<u>COVARIANT</u> <u>DIFFERENTIATION</u> <u>ALONG</u> α) SUCH THAT $\forall a, b \in \mathbb{R}$ $\forall X, Y \in T_\alpha(TM)$ $\forall f \in C^\infty(I)$,

1. $\dfrac{D}{dt}(aX + bY) = a\dfrac{DX}{dt} + b\dfrac{DY}{dt}$

2. $\dfrac{D}{dt}(fX) = f\dfrac{DX}{dt} + \dfrac{df}{dt}X$

3. IF, ON SOME OPEN SET U IN M, X IS INDUCED BY A VECTOR FIELD $\tilde{X}$ (I.E., $X(t) = \tilde{X}(\alpha(t))$), THEN

$$\frac{DX}{dt} = \nabla_{\alpha'(t)}\tilde{X}$$

AT EACH POINT WITH $\alpha(t) \in U$.

<u>NOTES</u>: $\nabla_{\alpha'(t)}\tilde{X}$ MAKES SENSE BECAUSE $\nabla_V W(p)$ DEPENDS ONLY ON THE VALUE OF V AT p AND THE VALUES OF W ON A NEIGHBORHOOD OF p. IT IS NOT OBVIOUS (PARTITION OF UNITY ARGUMENT), BUT <u>EVERY</u> $X \in T_\alpha(TM)$ IS LOCALLY INDUCED BY A VECTOR FIELD $\tilde{X}$.

PROOF OF THE THEOREM :

NEED TO PROVE EXISTENCE AND UNIQUENESS, BUT WE'LL DO THEM IN REVERSE ORDER. THUS, ASSUMING THE EXISTENCE OF THE OPERATOR $\frac{D}{dt}$ WE SHOW THAT IT IS UNIQUELY DETERMINED BY # 1-3.

FIX $X \in \mathcal{T}_\alpha(TM)$ AND CONSIDER $\frac{DX}{dt} \in \mathcal{T}_\alpha(TM)$. LET (U, φ) BE A CHART ON M WITH $\alpha(I) \cap U \neq \emptyset$ AND WITH COORDINATE FUNCTIONS $x^1, \dots, x^n$. ON U WE WRITE

$$X(t) = X^j(t) \frac{\partial}{\partial x^j}\Big|_{\alpha(t)}$$

WE CAN IDENTIFY $\frac{\partial}{\partial x^j}\Big|_{\alpha(t)}$ WITH THE VECTOR FIELD ALONG α THAT SENDS t TO $\frac{\partial}{\partial x^j}\Big|_{\alpha(t)}$ SO PROPERTIES # 1 AND 2 GIVE

$$\begin{aligned} \frac{DX}{dt} &= \frac{D}{dt}\left(X^j \frac{\partial}{\partial x^j}\Big|_{\alpha(t)}\right) \\ &= X^j \frac{D}{dt}\left(\frac{\partial}{\partial x^j}\Big|_{\alpha(t)}\right) + \frac{dX^j}{dt} \frac{\partial}{\partial x^j}\Big|_{\alpha(t)} \end{aligned}$$

BY # 3,

$$\frac{DX}{dt} = X^j \nabla_{\alpha'(t)}\left(\frac{\partial}{\partial x^j}\right) + \frac{dX^j}{dt} \frac{\partial}{\partial x^j}\Big|_{\alpha(t)}$$

NOW WRITE $(\varphi \circ \alpha)(t) = (x^1(t), \ldots, x^n(t))$ SO THAT

$$\alpha'(t) = \frac{dx^i}{dt} \frac{\partial}{\partial x^i}\Big|_{\alpha(t)}$$

SO THAT

$$\nabla_{\alpha'(t)} \left(\frac{\partial}{\partial x^j} \right) = \frac{dx^i}{dt} \nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial}{\partial x^j} \right) (\alpha(t))$$

$$= \frac{dx^i}{dt} \Gamma^k_{ij}(\alpha(t)) \frac{\partial}{\partial x^k}\Big|_{\alpha(t)}$$

THUS,

$$\frac{DX}{dt} = X^j \frac{dx^i}{dt} \Gamma^k_{ij}(\alpha(t)) \frac{\partial}{\partial x^k}\Big|_{\alpha(t)} + \frac{dX^k}{dt} \frac{\partial}{\partial x^k}\Big|_{\alpha(t)}$$

$$= \left(\Gamma^k_{ij}(\alpha(t)) \frac{dx^i}{dt} X^j + \frac{dX^k}{dt} \right) \frac{\partial}{\partial x^k}\Big|_{\alpha(t)}$$

$$\boxed{\frac{DX}{dt} = \left(\frac{dX^k}{dt} + \Gamma^k_{ij}(\alpha(t)) \frac{dx^i}{dt} X^j \right) \frac{\partial}{\partial x^k}\Big|_{\alpha(t)}}$$

SINCE THE RIGHT-HAND SIDE IS UNIQUELY DETERMINED BY α, X AND ∇ WE FIND THAT $\frac{DX}{dt}$ IS UNIQUELY DETERMINED ON ANY COORDINATE NEIGHBORHOOD INTERSECTING $\alpha(I)$ AND THEREFORE EVERYWHERE.

THE REASON FOR PROVING UNIQUESS FIRST IS THAT IT HAS YIELDED THIS EXPLICIT LOCAL FORMULA WHICH NOW TELLS US HOW TO

PROVE EXISTENCE. IN ANY CHART (U, φ) WITH $\alpha(I) \cap U \neq \emptyset$ WE DEFINE $\frac{DX}{dt}$ BY THE FORMULA IN THE BOX ON THE PREVIOUS PAGE.

EXERCISE : VERIFY PROPERTIES # 1-3 DIRECTLY FROM THIS FORMULA.

NOW, IF WE HAVE ANOTHER CHART (V, ψ) WITH $\alpha(I) \cap V \neq \emptyset$ AND DEFINE $\frac{DX}{dt}$ BY THE ANALOGOUS FORMULA ON V, AND IF $U \cap V \neq \emptyset$, THEN THE DEFINITIONS AGREE ON THIS INTERSECTION BY THE UNIQUENESS WE HAVE ALREADY PROVED. $\square$

A VECTOR FIELD X ALONG α IS SAID TO BE PARALLEL (WITH RESPECT TO ∇) IF $\frac{DX}{dt} = 0$ FOR ALL $t \in I$, I.E., IF

$$\frac{dX^k}{dt} + \Gamma^k_{ij}(\alpha(t)) \frac{dx^i}{dt} X^j = 0$$

$$k = 1, \dots, n$$

A CURVE α FOR WHICH THE TANGENT FIELD α' IS PARALLEL ALONG α IS CALLED A GEODESIC (OF THE CONNECTION ∇)

IN THIS CASE, $X^k = \frac{dx^k}{dt}$ SO THE PREVIOUS SET OF DIFFERENTIAL EQUATIONS BECOMES

$$\frac{d^2x^k}{dt^2} + \Gamma^k_{ij}(\alpha(t))\frac{dx^i}{dt}\frac{dx^j}{dt} = 0$$

$$k = 1, \ldots, n$$

A FEW GENERAL REMARKS ON PARALLEL TRANSLATION AND GEODESICS:

1. IF THE MANIFOLD M, CONNECTION ∇ AND CURVE α ARE GIVEN, THEN, IN ANY COORDINATE NEIGHBORHOOD,

$$\frac{dX^k}{dt} + \Gamma^k_{ij}(\alpha(t))\frac{dx^i}{dt}X^j = 0$$

$$k = 1, \ldots, n$$

IS A 1^{ST} ORDER, LINEAR SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS FOR $X^1, \ldots, X^n$.

THE EXISTENCE AND UNIQUENESS THEOREM FOR SUCH SYSTEMS IMPLIES:

GIVEN A $t_0 \in I$ AND A TANGENT VECTOR $X_0 \in T_{\alpha(t_0)}(M)$ $\exists$ UNIQUE VECTOR FIELD $X \in T_\alpha(TM)$ THAT IS PARALLEL ALONG α AND SATISFIES $X(t_0) = X_0$

FOR EACH $t \in I$, $X(t)$ IS CALLED THE PARALLEL TRANSLATION OF $X_0 \in T_{\alpha(t_0)}(M)$ TO $T_{\alpha(t)}(M)$.

THIS DEFINES A MAP

$$P = P_{t_0}^{t}(\alpha) : T_{\alpha(t_0)}(M) \to T_{\alpha(t)}(M)$$

CALLED PARALLEL TRANSLATION ALONG α FROM $\alpha(t_0)$ TO $\alpha(t)$ (WITH RESPECT TO THE CONNECTION ∇).

EXERCISE: SHOW THAT PARALLEL TRANSLATION IS A LINEAR ISOMORPHISM.

THUS, A CONNECTION ON M PROVIDES A MEANS OF "CONNECTING" TANGENT SPACES AT DIFFERENT POINTS.

2. IF THE MANIFOLD M AND CONNECTION ∇ ARE GIVEN, THEN, IN ANY COORDINATE NEIGHBORHOOD,

$$\frac{d^2x^k}{dt^2} + \Gamma_{ij}^{k}(x^1,\dots,x^n)\frac{dx^i}{dt}\frac{dx^j}{dt} = 0 \qquad k = 1,\dots,n$$

IS A 2^{ND} ORDER, NONLINEAR SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS.

THE EXISTENCE AND UNIQUENESS THEOREM FOR SUCH SYSTEMS IMPLIES :

GIVEN $p \in M$ AND $v_p \in T_p(M)$ THERE EXISTS A (MAXIMAL) INTERVAL I ABOUT 0 IN $\mathbb{R}$ AND A UNIQUE GEODESIC

$$\alpha : I \longrightarrow M$$

SATISFYING

$$\alpha(0) = p$$
$$\alpha'(0) = v_p$$

(α IS THE UNIQUE MAXIMAL GEODESIC THAT <u>FITS</u> v_p AT p)

THE ESSENTIAL POINT HERE IS THAT EXISTENCE IS ONLY <u>LOCAL</u> IN GENERAL.

<u>EXERCISE</u> : FIND ALL OF THE GEODESICS FOR THE STANDARD CONNECTION ON $\mathbb{R}^n$.

TO FIND MORE INTERESTING EXAMPLES WE NEED A SUPPLY OF MORE INTERESTING CONNECTIONS AND WE WILL TURN TO THIS SOON.