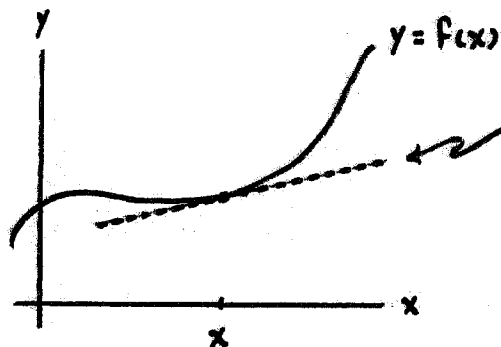


## PARTIAL DERIVATIVES

RECALL: GIVEN  $y = f(x)$  AND  $x$  IN ITS DOMAIN.



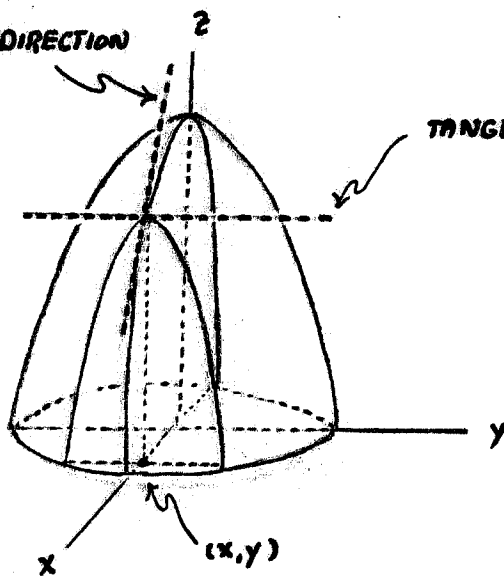
$$\text{SLOPE} = f'(x) = \frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

= MEASURE OF THE RATE AT WHICH  
F IS CHANGING AT X

NOW SUPPOSE  $z = f(x, y)$  AND  $(x, y)$  IS A POINT IN ITS DOMAIN.

"RATE AT WHICH F IS CHANGING AT  $(x, y)$ "  
MAKES NO SENSE SINCE F CAN CHANGE AT  
DIFFERENT RATES IN DIFFERENT DIRECTIONS  
AT  $(x, y)$ , E.G.,

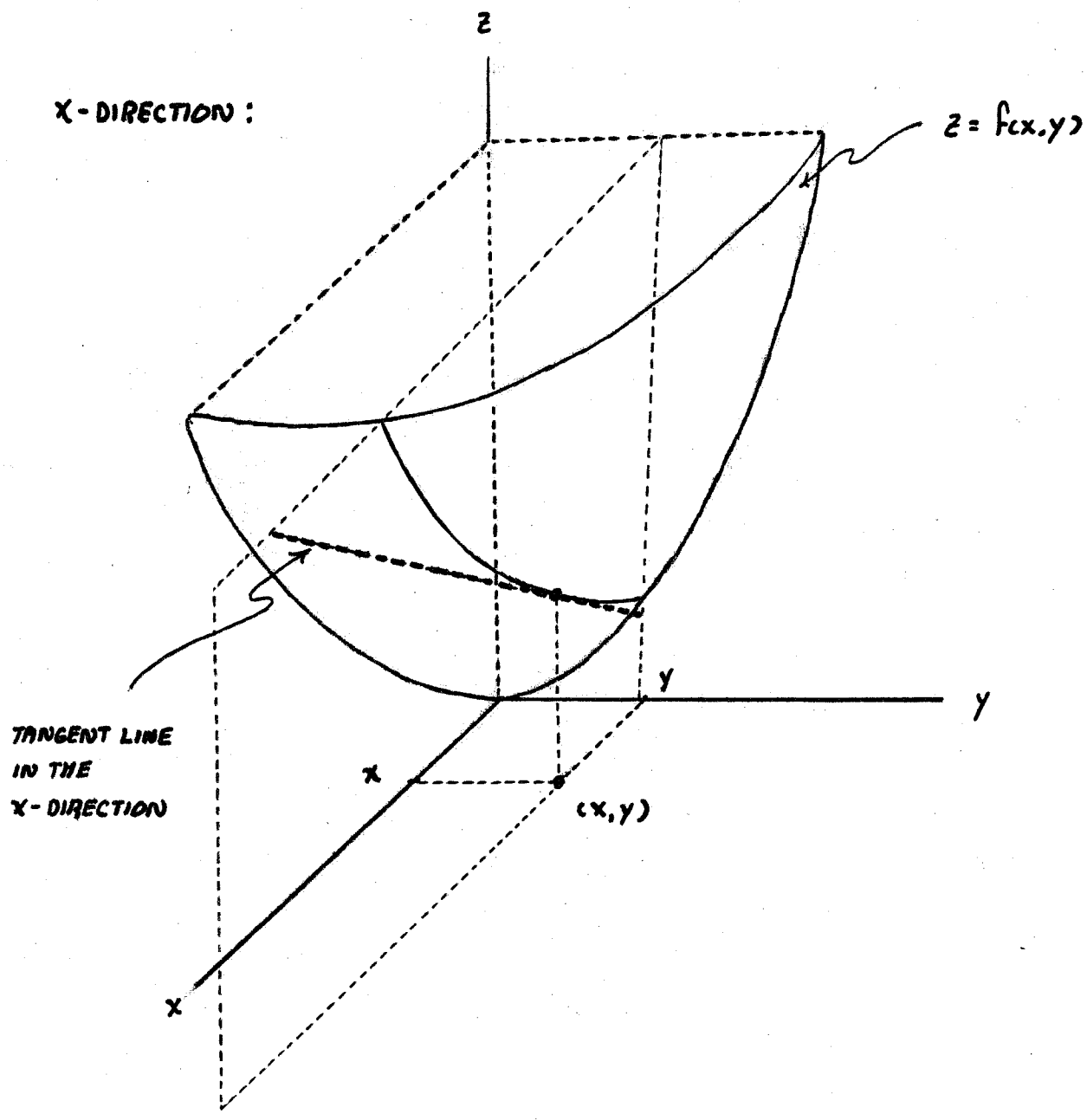
TANGENT LINE IN THE X-DIRECTION



TANGENT LINE IN THE Y-DIRECTION

WE WILL BEGIN BY COMPUTING RATES OF CHANGE IN THE X-DIRECTION AND IN THE Y-DIRECTION.

X-DIRECTION :

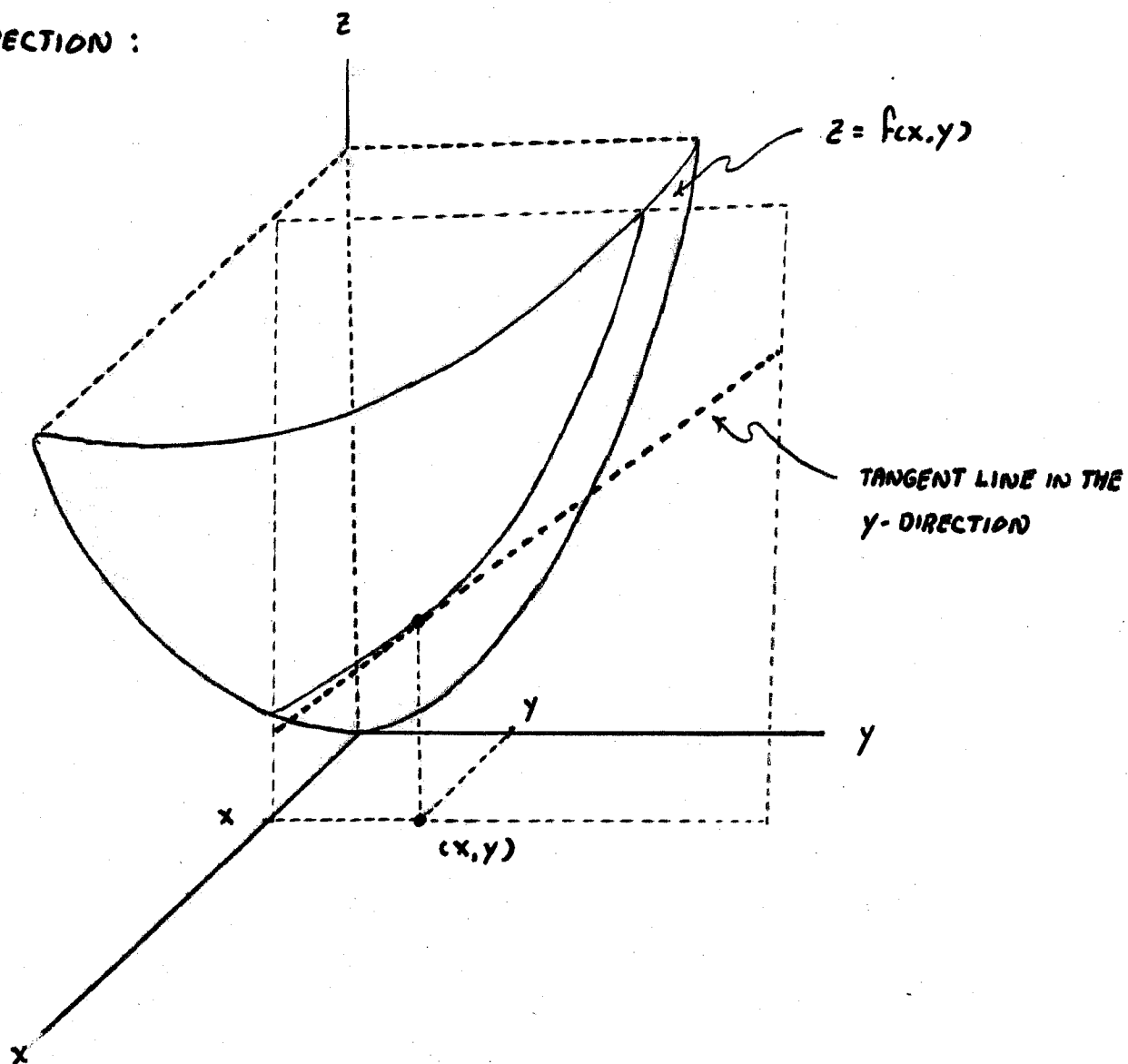


SLOPE OF TANGENT LINE IN X-DIRECTION = PARTIAL DERIVATIVE OF F  
WITH RESPECT TO X

$$= \frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

(HOLD y FIXED AND DIFFERENTIATE WITH RESPECT TO x AS USUAL)

y-DIRECTION :



SLOPE OF TANGENT LINE IN y-DIRECTION = PARTIAL DERIVATIVE OF f  
WITH RESPECT TO y

$$= \frac{\partial f}{\partial y} = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

(HOLD x FIXED AND DIFFERENTIATE  
 WITH RESPECT TO y AS USUAL)

SOME ALTERNATE NOTATIONS :  $z = f(x, y)$

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} (f(x, y)) = z_x = D_x f = D_1 f = \dots$$

$$\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} (f(x, y)) = z_y = D_y f = D_2 f = \dots$$

EXAMPLES :

1.  $z = f(x, y) = 2x + 5x^4 y^3$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (2x + 5x^4 y^3) = 2 + (20x^3) y^3 = 2 + 20x^3 y^3$$

$$z_y = \frac{\partial}{\partial y} (2x + 5x^4 y^3) = 0 + (5x^4)(3y^2) = 15x^4 y^2$$

$$\frac{\partial f}{\partial x} (1, 2) = 2 + 20(1^3)(2^3) = 162$$

$$z_y (1, 2) = 15(1^4)(2^2) = 60$$

2.  $z = f(x, y) = (x^2 + y^2) e^{x^2 y}$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} ((x^2 + y^2) e^{x^2 y})$$

$$= (x^2 + y^2) \frac{\partial}{\partial x} (e^{x^2 y}) + e^{x^2 y} \frac{\partial}{\partial x} (x^2 + y^2)$$

$$= (x^2 + y^2) e^{x^2 y} \frac{\partial}{\partial x} (x^2 y) + e^{x^2 y} (2x + 0)$$

$$= 2xy(x^2 + y^2) e^{x^2 y} + 2x e^{x^2 y}$$

$$3. R(s, t) = s^4 \sin(st^3)$$

$$\begin{aligned} \frac{\partial R}{\partial s} &= \frac{\partial}{\partial s} (s^4 \sin(st^3)) \\ &= s^4 \frac{\partial}{\partial s} (\sin(st^3)) + \sin(st^3) \frac{\partial}{\partial s} (s^4) \\ &= s^4 \cos(st^3) \frac{\partial}{\partial s} (st^3) + 4s^3 \sin(st^3) \\ &= s^4 t^3 \cos(st^3) + 4s^3 \sin(st^3) \end{aligned}$$

4. FIND  $\frac{\partial z}{\partial x}$  BY IMPLICIT DIFFERENTIATION IF

$$x \sin(y+z) - xy + z^2 = 0.$$

(z IS THE FUNCTION, x THE INDEPENDENT VARIABLE AND y IS HELD FIXED)

$$\frac{\partial}{\partial x} (x \sin(y+z) - xy + z^2) = \frac{\partial}{\partial x} (0)$$

$$\frac{\partial}{\partial x} (x \sin(y+z)) - \frac{\partial}{\partial x} (xy) + \frac{\partial}{\partial x} (z^2) = 0$$

$$x \frac{\partial}{\partial x} \sin(y+z) + \sin(y+z) \frac{\partial}{\partial x} (x) - y + 2z \frac{\partial z}{\partial x} = 0$$

$$x \cos(y+z) \frac{\partial}{\partial x} (y+z) + \sin(y+z) - y + 2z \frac{\partial z}{\partial x} = 0$$

$$x \cos(y+z) \left(0 + \frac{\partial z}{\partial x}\right) + \sin(y+z) - y + 2z \frac{\partial z}{\partial x} = 0$$

$$[x \cos(y+z) + 2z] \frac{\partial z}{\partial x} = y - \sin(y+z)$$

$$\frac{\partial z}{\partial x} = \frac{y - \sin(y+z)}{x \cos(y+z) + 2z}$$



EXAMPLE :  $z = f(x, y) = 2x + 5x^4 y^3$

$$z_x = 2 + 20x^3 y^3 \quad : \quad z_{xx} = 60x^2 y^3 \quad z_{xy} = 60x^3 y^2$$

$$z_y = 15x^4 y^2 \quad : \quad z_{yy} = 30x^4 y \quad z_{yx} = 60x^3 y^2$$

NOTE :  $z_{xy} = z_{yx}$

THIS IS NO ACCIDENT :

THEOREM : LET  $z = f(x, y)$ . IF  $z_{xy}$  AND  $z_{yx}$  ARE BOTH CONTINUOUS,

THEN

$$z_{xy} = z_{yx}$$

(I. E.,

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} )$$

ALL OF THIS EXTENDS EASILY TO FUNCTIONS OF MORE THAN TWO VARIABLES, E.G.,

$w = z \text{ ARCTAN} \left( \frac{y}{x} \right) :$

$$\frac{\partial w}{\partial x} = z \frac{\partial}{\partial x} \text{ARCTAN} \left( \frac{y}{x} \right)$$

$$= z \left( \frac{1}{1 + \left( \frac{y}{x} \right)^2} \right) \frac{\partial}{\partial x} \left( \frac{y}{x} \right)$$

$$= z \left( \frac{1}{1 + \frac{y^2}{x^2}} \right) \left( -\frac{y}{x^2} \right) = -\frac{yz}{x^2 + y^2}$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z} (2 \operatorname{ARCTAN}(\frac{y}{x})) = \operatorname{ARCTAN}(\frac{y}{x})$$

ONE MORE EXAMPLE : SHOW THAT  $\mu(x,y) = \ln \sqrt{x^2+y^2}$  SATISFIES

$$\mu_{xx} + \mu_{yy} = 0$$

(THE "2-DIMENSIONAL LAPLACE EQUATION").

$$\mu = \ln \sqrt{x^2+y^2} = \frac{1}{2} \ln(x^2+y^2) \Rightarrow$$

$$\mu_x = \frac{1}{2} \frac{1}{x^2+y^2} (2x) = \frac{x}{x^2+y^2}$$

$$\begin{aligned} \mu_{xx} &= \frac{(x^2+y^2) \frac{\partial}{\partial x}(x) - x \frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2} \\ &= \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2} \end{aligned}$$

SIMILARLY,

$$\mu_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

SO

$$\begin{aligned} \mu_{xx} + \mu_{yy} &= \frac{-x^2+y^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} \\ &= \frac{-x^2+y^2+x^2-y^2}{(x^2+y^2)^2} \\ &= \frac{0}{(x^2+y^2)^2} = 0 \end{aligned}$$